

Ch.4 Binomial Theorem for Positive Integral Indices Solution 二項式定理題解

QA1 解

$$\begin{aligned} \text{第 4 項 : 第 5 項} &= C_3^9(3x)^{9-3}(4y)^3 : C_3^9(3x)^{9-4}(4y)^4 \\ &= 84(3x) : 126(4y) \\ &= x : 2y \end{aligned}$$

QA2 解

$$\begin{aligned} \left(1 - \frac{1}{2x^2}\right)^{12} &= 1 - 12\left(\frac{1}{2x^2}\right) + 66\left(\frac{1}{2x^2}\right)^2 - \dots \\ &= 1 - \frac{6}{x^2} + \frac{33}{2x^4} - \dots \end{aligned}$$

QA3 解

$$\begin{aligned} \text{第 } (r+1) \text{ 項} &= C_r^6(x^2)^{6-r}\left(\frac{-2}{x}\right)^r \\ &= C_r^6(-2)^r x^{12-3r} \end{aligned}$$

若這一項不含 x ，則

$$\begin{aligned} \therefore 12 - 3r &= 0 \\ r &= 4 \end{aligned}$$

$$\begin{aligned} \text{由此，所求的項} &= C_4^6(-2)^4 \\ &= 240 \end{aligned}$$

QA4 解

$$\begin{aligned} (1-3x)^4(1+x)^7 &= [1-4(3x)+6(3x)^2-4(3x)^3+(3x)^4][1+7x+21x^2+35x^3+\dots] \\ &= (1-12x+54x^2-108x^3+\dots)(1+7x+21x^2+35x^3+\dots) \\ \therefore x^3 \text{ 的係數} &= -108+378-252+35 \\ &= 53 \end{aligned}$$

QA5 解

$$\begin{aligned} (1-x+x^2)^n &= 1-nx(1-x) + \frac{n(n-1)}{2}x^2(1-x)^2 - \dots \\ &= 1-nx + \left[n + \frac{n(n-1)}{2}\right]x^2 + \dots \end{aligned}$$

依題意，

$$\begin{aligned} n + \frac{n(n-1)}{2} &= 36 \\ n &= 8 \text{ 或 } -9 \text{ (捨去)} \\ \therefore n &= 8 \end{aligned}$$

QA6 解

$$\begin{aligned} \text{(a)} \quad (1+ax)^4(1-2x)^5 &= (1+4ax+6a^2x^2+\dots)[1-5(2x)+10(2x)^2-\dots] \\ &= (1+4ax+6a^2x^2+\dots)(1-10x+40x^2-\dots) \\ &= 1+(4a-10)x+(6a^2-40a+40)x^2+\dots \end{aligned}$$

(b) 依題意，

$$\begin{aligned} 4a-10 &= 0 \\ \therefore a &= \frac{5}{2} \end{aligned}$$

$$\therefore x^2 \text{ 的係數} = 6\left(\frac{5}{2}\right)^2 - 40\left(\frac{5}{2}\right) + 40 = -\frac{45}{2}$$

QA7 解

$$\frac{C_3^8}{C_2^{12} \times C_1^5} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{1}{\frac{12 \times 11}{2 \times 1} \times 5} = \frac{28}{165}$$

QA8 解

$$\begin{aligned} C_{r+1}^{20} &= C_{3r-5}^{20} \\ r+1 &= 3r-5 \quad \text{或} \quad r+1 = 20-(3r-5) \\ r &= 3 \quad \text{或} \quad r = 6 \end{aligned}$$

QA9 解

$$\begin{aligned} \text{(a)} \quad (1+3x)^4 &= 1+4(3x)+6(3x)^2+4(3x)^3+(3x)^4 \\ &= 1+12x+54x^2+108x^3+81x^4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{取 } x=0.01, \\ 1.03^4 &= [1+3(0.01)]^4 \\ &= 1+12(0.01)+54(0.01)^2+108(0.01)^3+81(0.01)^4 \\ &= 1+0.12+0.0054+0.000108+0.0000081 \\ &= 1.12550881 \end{aligned}$$

QA10 解

$$\begin{aligned} (2x^3-1)^5 &= (2x^3)^5 + 5(2x^3)^4(-1) + 10(2x^3)^3(-1)^2 + 10(2x^3)^2(-1)^3 + \\ &\quad 5(2x^3)(-1)^4 + (-1)^5 \\ &= 32x^{15} - 80x^{12} + 80x^9 - 40x^6 + 10x^3 - 1 \end{aligned}$$

QA11 解

$$\begin{aligned} \left(2 - \frac{1}{3}x\right)^6 (1+x)^7 &= \left[2^6 + C_1^6(2^5)\left(-\frac{1}{3}x\right) + C_2^6(2^4)\left(-\frac{1}{3}x\right)^2 + C_3^6(2^3)\left(-\frac{1}{3}x\right)^3 + \dots\right] \\ &\quad \times (1 + C_1^7x + C_2^7x^2 + C_3^7x^3 + \dots) \\ &= \left(64 - 64x + \frac{80}{3}x^2 - \frac{160}{27}x^3 + \dots\right) \times \\ &\quad (1 + 7x + 21x^2 + 35x^3 + \dots) \\ \therefore x^3 \text{ 的係數} &= (64)(35) + (-64)(21) + \left(\frac{80}{3}\right)(7) + \left(-\frac{160}{27}\right)(1) \\ &= 1076\frac{20}{27} \end{aligned}$$

QA12 解

$$\begin{aligned} (1-x)^7 \left(1 + \frac{2}{x}\right)^3 &= (1-7x+21x^2-35x^3+35x^4-21x^5+\dots) \\ &\quad \times \left(1 + \frac{6}{x} + \frac{12}{x^2} + \frac{8}{x^3}\right) \\ \text{所求的項} &= [(21)(1) + (-35)(6) + (35)(12) + (-21)(8)]x^2 \\ &= 63x^2 \end{aligned}$$

QA13 解

$$\begin{aligned} (1-3x+2x^2)^4 &= [1-x(3-2x)]^4 \\ &= 1-4x(3-2x)+6x^2(3-2x)^2+\dots \\ &= 1-12x+8x^2+54x^2+\dots \\ &= 1-12x+62x^2+\dots \end{aligned}$$

QB1 解

(a) $(1+ax)^2(1+bx)^7 = (1+2ax+a^2x^2)(1+7bx+21b^2x^2+\dots)$
 $= 1+(2a+7b)x+(a^2+14ab+21b^2)x^2+\dots$

(b) (i) $2a+7b = -\frac{5}{2}$ (1)

$a^2+14ab+21b^2 = -\frac{27}{4}$ (2)

由 (1), $a = -\frac{7b}{2}-\frac{5}{4}$ (3)

把 (3) 代入 (2),

$(-\frac{7b}{2}-\frac{5}{4})^2 + 14(-\frac{7b}{2}-\frac{5}{4})b + 21b^2 = -\frac{27}{4}$

$\frac{49b^2}{4} + \frac{35b}{4} + \frac{25}{16} - 49b^2 - \frac{35}{2}b + 21b^2 = -\frac{27}{4}$
 $36b^2 + 20b - 19 = 0$

$\therefore b = \frac{1}{2}$ 或 $-\frac{19}{18}$ (4)

把 (4) 代入 (3),

當 $b = \frac{1}{2}$, $a = -\frac{7}{2}(\frac{1}{2}) - \frac{5}{4} = -3$;

當 $b = -\frac{19}{18}$, $a = -\frac{7}{2}(-\frac{19}{18}) - \frac{5}{4} = \frac{22}{9}$ 。

即 $(a=-3$ 及 $b = \frac{1}{2})$ 或 $(a = \frac{22}{9}$ 及 $b = -\frac{19}{18})$ 。

(ii) $(1+ax)^3(1+bx)^7 = (1+ax)[(1+ax)^2(1+bx)^7]$
 $= (1+ax)(1-\frac{5}{2}x-\frac{27}{4}x^2+\dots)$

$\therefore x$ 的係數 = $a - \frac{5}{2}$

當 $a=-3$,

x 的係數 = $-3 - \frac{5}{2} = -\frac{11}{2}$

當 $a = \frac{22}{9}$,

x 的係數 = $\frac{22}{9} - \frac{5}{2} = \frac{1}{18}$

QB2 解

設 $S(n)$ 為命題

“ $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = 1^4 + 2^4 + \dots + n^4$ ” ,

當 $n=1$,

左方 = $\frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30} = 1$

右方 = 1

$\therefore S(1)$ 成立。

假設 $S(k)$ 成立。

即 $\frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} = 1^4 + 2^4 + \dots + k^4$ 。

則 $\frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{(k+1)}{30}$
 $= \frac{1}{5}(k^5+5k^4+10k^3+10k^2+5k+1) +$
 $\frac{1}{2}(k^4+4k^3+6k^2+4k+1) + \frac{1}{3}(k^3+3k^2+3k+1) -$
 $\frac{k}{30} - \frac{1}{30}$

$= \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + (k^4+4k^3+6k^2+4k+1)$

$= 1^4+2^4+\dots+k^4 + (k+1)^4$

$\therefore S(k+1)$ 成立。

根據數學歸納法, $S(n)$ 對所有正整數 n 皆成立。

QB3 解

(a) $(1+px+x^2)^n = 1+nx(p+x) + \frac{n(n-1)}{2}x^2(p+x)^2+\dots$

$= 1+npix + \left[n + \frac{n(n-1)}{2}p^2 \right]x^2+\dots$

(b) 依題意,

$np = 2$ (1)

$n + \frac{n(n-1)}{2}p^2 = \frac{59}{5}$ (2)

由 (1), $p = \frac{2}{n}$ (3)

把 (3) 代入 (2),

$n + \frac{n(n-1)}{2} \left(\frac{2}{n}\right)^2 = \frac{59}{5}$

$\therefore 5n^2 - 49n - 10 = 0$

$\therefore n = 10$ 或 $-\frac{1}{5}$ (捨去)

$\therefore n = 10$ 及 $p = \frac{1}{5}$

(c) $1 + \frac{1}{5}x + x^2 = 1.0044$

$\therefore 5x^2 + x - 0.022 = 0$
 $x = 0.02$ 或 -0.22