Ch.14 Limits and Derivatives and Differentiation 函數的極限與導數及微分法

A1. Find 求 $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$。 (4 分)

A2. Find 求 $\lim_{x \to -1} \frac{1 - |x|}{1 + x}$。 (4 分)

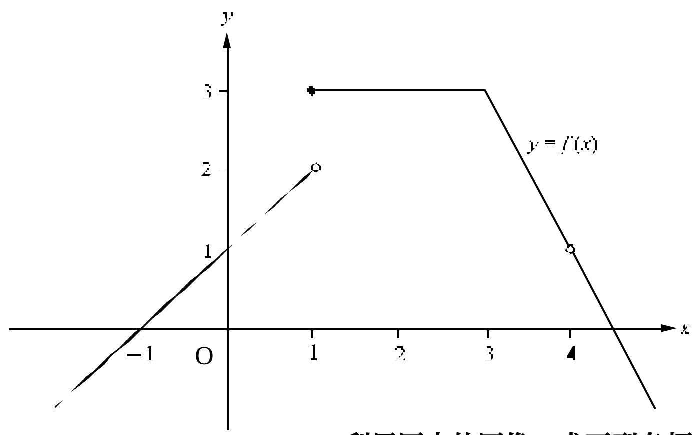
A3. Find 求 $\lim_{x \to 0} \frac{x}{|x|} \sin x$。 (6 分)

A4. Find 求 $\lim_{x \to \infty} \frac{x^3 + 2x - 3}{4x^3 - 8x^2 + 6}$。 (4 分)

A5. Find the derivative of $y = 2\sqrt{x}$ from first principles. **試從基本原理求** $y = 2\sqrt{x}$ **的導數。** (5 分)

A6. Find the derivative of $y = \frac{t+1}{t-1}$ from first principles. **試從基本原理求** $y = \frac{t+1}{t-1}$ **的導數。** (5 分)

A7.



Use the graph in the figure to find the following limits. **利用圖中的圖像，求下列各極限。**

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3} f(x)$ (c) $\lim_{x \to 4} f(x)$ (5 分)

A8. Find **試求** $\lim_{x \to 0} \frac{(k+x)^3 - k^3}{x}$， where k is a constant **其中** k **為一常數。** (5 分)

A9. Find 求 $\lim_{x \to 0} \frac{\sin^3 3x}{x \sin^2 5x}$。 (5 分)

A10. Find 求 $\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$。 (6 分)

A11. Find 求 $\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 8}{(x+1)(5x-3)^2}$。 (5 分)

A12. Let 設 $y = f(x) = 2x^2 - 3x + 7$ and 及 $x_1 = 2$。

(a) Complete the table below. **試完成下表。**

Δx	1	0.1	0.01	0.001	0.000 1
$x_1 + \Delta x$	3	2.1			
$\Delta y = f(x_1 + \Delta x) - f(x_1)$					
$\frac{\Delta y}{\Delta x}$					

(b) Find 求 $\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$。 (8 分)

A13. Let 設 $y = f(x) = (x+5)^2$ 及 $\Delta y = f(x + \Delta x) - f(x)$。

(a) Find **試求** Δy。

(b) Find **試求** $\frac{\Delta y}{\Delta x}$。

(c) Hence **由此，** find from first principles **從基本原理求** $\frac{dy}{dx}$。 (6 分)

A14. Let 設 $f(x) = 2x^3 + 7x$。

(a) Find from first principles **從基本原理求** $f'(x)$。

(b) Evaluate and interpret **試計算及解釋** $f'(-1)$。 (8 分)

A15. (a) Given 已知 $y = 20t + t^2$, find from first principles 從基本原理求 $\frac{dy}{dt}$。

(b) Suppose the population of certain bacteria grows in such a way that at time t hours there are $20t + t^2$ bacteria. Find the rate of growth at time $t = 10$ hours.
假設某細菌在 t 小時內的增長為 $20t + t^2$ 。試求當 $t = 10$ 小時，該細菌的增長率。 (7 分)

A16. If 若 $y = \sqrt[4]{\frac{x+1}{2x+3}}$, find 試求 $\frac{dy}{dx}$。 (4 分)

A17. Find the slope of the tangent to the curve $x^2 + xy + y^5 = 3$ at the point (1, 1).
試求曲線 $x^2 + xy + y^5 = 3$ 在點 (1, 1) 的切線斜率。 (5 分)

A18. If 若 $y = \sqrt[3]{\operatorname{cosec}^2 2x}$, find 試求 $\frac{dy}{dx}$ 的值。 (4 分)

A19. If 若 $f(x) = x\sin\frac{\pi}{x}$, find 試求 $f''(6)$ 的值。 (6 分)

A20. It is given that 已知 $x = 2\sec\theta$ and 及 $y = 3\tan\theta$。at 當 $\theta = \frac{\pi}{4}$ 時，Find 試求 $\frac{dy}{dx}$ 的值。 (5 分)

A21. 若 $x = y + \cos y$，find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of y 試以 y 表 $\frac{dy}{dx}$ 及 $\frac{d^2y}{dx^2}$ 的值。 (6 分)

A22. If 若 $f(x) = \frac{6x^2}{\sqrt{x^3}}$, find 試求

(a) $f'(x)$, (b) $f'(4)$。 (4 分)

A23. The gradient of the curve 曲線 $y = \frac{k}{x}$ at 在 $x = 1$ 時的斜率 is 為 -2。Find 試求

(a) the value of the constant 常數 k 的值,

(b) the equation of the tangent to the curve at $x = 1$ 此曲線在 $x = 1$ 時的切線方程,

(c) the points on the curve at which the gradient is $-\frac{1}{2}$ 此曲線斜率為 $-\frac{1}{2}$ 的點。 (8 分)

A24. Find the derivative of 試求 $y = (\sqrt[3]{x} - 4)(x^2 + 3x)$ 的導數。 (5 分)

A25. The equation of a curve is $y = x^3 - \frac{5}{2}x^2 + 2x + 1$. Find the points on the curve at which the tangent line is horizontal.

一曲線的方程為 $y = x^3 - \frac{5}{2}x^2 + 2x + 1$。若該曲線在某點有水平切線，求該點的坐標。 (8 分)

A26. If 若 $y = \frac{t^{-2} + 3}{t^{-1} - 2}$, at 當 $t = 2$ 時，find 試求 $\frac{dy}{dt}$ 的值。 (6 分)

A27. If 若 $y = \frac{\sqrt{x}}{(2x+1)(x-3)}$, find 試求 $\frac{dy}{dx}$ 的值。 (6 分)

A28. If 若 $y = (\sqrt{x} + 1)(\sqrt{x} - 1)^{\frac{3}{2}}$, find 試求 $\frac{dy}{dx}$ 的值。 (5 分)

A29. If 若 $F(t) = \left(\frac{3t+4}{2t-7}\right)^5$, find 試求 $F'(t)$ 的值。 (5 分)

A30. If 若 $x^3 - x^2y + 3xy^2 + y^3 = 0$, find 試求 $\frac{dy}{dx}$ 的值。 (5 分)

A31. If 若 $y = \cos\left(\frac{2x+3}{2x-3}\right)$, find 試求 $\frac{dy}{dx}$ 的值。 (5 分)

A32. Let 設 $y = x\cos 7x$。

(a) Find 求 $\frac{dy}{dx}$ and 和 $\frac{d^2y}{dx^2}$。

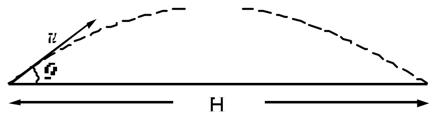
(b) Hence 由此，find 求 $\frac{d^2y}{dx^2} + 49y$。 (6 分)

A33. Let 設 $f(x) = \frac{\tan x}{\sqrt{x^2 + A}}$, where 其中 A 為一 constant 常數。If 若 $f'(0) = \frac{1}{2}$, find 試求 A 的數值。 (5 分)

A34. The parametric equations of a curve are 一曲線之參數方程為
$x = 2\cos t + \cos 2t$ and 及 $y = \sin 2t - 2\sin t$。Show that 證明 $\frac{dy}{dx} = \tan\frac{t}{2}$。 (8 分)

A35. If 若 $y=\sqrt{\dfrac{3+2x^2}{5-x^2}}$ and 及 $\dfrac{dy}{dx}=y\left(\dfrac{\mathrm{A}x}{3+2x^2}+\dfrac{\mathrm{B}x}{5-x^2}\right)$，find 試求常數 A 和 B 的值。 (6 分)

A36.



When a ball is projected at an elevation θ with a velocity u, the horizontal range H is given by
當球以初速 u 拋出，而 u 與水平所成的仰角為 θ 時，其水平位移為

$$\mathrm{H} = \frac{u^2 \sin 2\theta}{g}。$$

If $u = 20$, $g = 10$, and the angle θ is adjusted to $\dfrac{\pi}{6}$ with a possible error of 0.005 radian, find the possible error in H using differentials.

若 $u = 20$，$g = 10$，而角 θ 調校至 $\dfrac{\pi}{6}$，其弧的可能偏差為 0.005 弧度。試利用微分求 H 的可能偏差。(4 分)

A37. Find the equation of the tangent with the least gradient to the curve $y = x^3 + x^2 + x + 1$.
求在曲線 $y = x^3 + x^2 + x + 1$ 上，斜率最小的切線的方程。 (7 分)

A38. (a) Find the turning point of the graph of $y = 2\sqrt{x} - x$.
試求 $y = 2\sqrt{x} - x$ 的圖像的轉向點。

(b) Sketch the graph for $0 \le x \le 5$.
描繪此函數在 $0 \le x \le 5$ 的圖像。 (8 分)

A39. A ball is thrown vertically upward from a point P so that its height at time t seconds is $h = (30t - 5t^2)$ metres. Q is another point on the same level as P and is 30 m from it. At time $t = 4$, find the rate of change of the angle of elevation θ of the stone from Q.
一球由點 P 向上拋，球在時間 t 秒的高度為 $h = (30t - 5t^2)$ m。點 Q 與 P 在同一水平上，而 Q 和 P 相距 30 m。當 $t = 4$，求由 Q 向球的仰角的變率。 (8 分)

A40. A right-angled triangle has hypotenuse 10 cm long and one of its acute angle is θ.
一直角三角形的斜邊長 10 cm，而其中一銳角為 θ。

(a) Express the lengths of the other two sides in terms of θ. 以 θ 表其餘兩邊的長。

(b) Find the maximum area of the triangle. 求三角形的最大面積。 (8 分)

A41. A particle P moves in a straight line and passes through a fixed point O. The distance s metres of P from O at time t seconds, where $t \ge 0$, is given by $s = 48t - t^3$. Calculate the acceleration of the particle when it is instantaneously at rest.
一質點 P 以直線移動，並經過一定點 O。在時間 t s ($t \ge 0$)，P 和 O 相距 s m，而 s 定義為 $s = 48t - t^3$。計算當該質點靜止時的瞬時加速度。 (6 分)

A42. Find the equations of the two tangents to the curve $y^2 = x^2y + 5$ at the points where $x = 2$.
求曲線 $y^2 = x^2y + 5$ 在 $x = 2$ 的兩切線的方程。 (8 分)

A43. An object moves around the circle $x^2 + y^2 = 100$. Its component of velocity in x-direction is $\dfrac{dx}{dt} = 2y$. Find $\dfrac{dy}{dt}$.

一物件繞著圓 $x^2 + y^2 = 100$ 移動。在 x 的方向，速度的分量是 $\dfrac{dx}{dt} = 2y$。求 $\dfrac{dy}{dt}$。 (4 分)

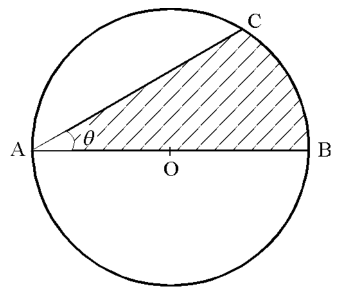
A44. A building is 20 m high. Its shadow on level ground is 25 m long. If the angle of elevation of the sun is decreasing at a rate of 15° per hour, at what rate is the shadow lengthening?
一建築物高 20 m，其在地面上的影子長 25 m。若太陽的仰角以每小時 15° 的速率遞減，問影子延長的速率為何？ (8 分)

A45. The equation of a curve is 一曲線的方程為 $y = x^4 - 4x^3$。

(a) Find the stationary points of the curve. 求曲線的駐點。

(b) Sketch the curve for $-1 \le x \le 4$. 描繪曲線在 $-1 \le x \le 4$ 的圖像。 (8 分)

A46.

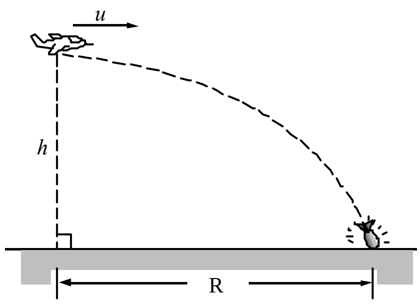


In the figure, AB is a diameter of the circle with centre O and radius 6 cm. C is a point on the circle such that ∠BAC = q.
圖中所示為一圓心為 O **而半徑為** 6 cm **的圓。**AB **為圓的直徑。**C **為圓上的一點，而** ∠BAC = q。

(a) Express the area S in cm^2 of the shaded region bounded by AB, AC and arc BC in terms of q.
試以 q **表由** AB、AC **及弧** BC **所圍成的陰影部分的面積** S (**單位為** cm^2)。

(b) If q is measured as 45° with a possible error of 0.5°, find the possible error in calculating S using differential.
若 q **量度為** 45°，**而可能誤差為** 0.5°，**試利用微分，求計算** S **時的可能誤差。** (8 **分**)

A47.



When a bomber is flying with a horizontal velocity u m/s at an altitude h m above the ground, the bomb that launched will hit the ground with a horizontal range R m given by $R = u\sqrt{\frac{2h}{g}}$, where g is the acceleration due to gravity.

當一轟炸機以水平速度 u m/s **離地面** h m **飛行時，投出的炸彈以水平位移** R m **打中地面，而** $R = u\sqrt{\frac{2h}{g}}$**，** w

其中 g **是重力加速度。**

(a) Express the differential dR in terms of the differentials du and dh.
試以微分 du **及** dh **表微分** dR 。

(b) If u and h are measured with possible errors of 1% and 2% respectively, what is the possible percentage error in computing R.
若 u **及** h **量度時的可能誤差分別為** 1% **及** 2% **，問計算** R **時的可能百分誤差為何？** (8 **分**)

B1. (a) Prove, by mathematical induction, that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$, for all positive integers n.

利用數學歸納法，證明對於所有正整數 n，$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$。 (7 **分**)

(b) Hence **由此，** or otherwise **或用其他方法，** find **求** $\lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\ \ldots\ +\frac{n^2}{n^3}\right)$。 (5 **分**)

B2. In a chemical reaction, the mass M, in grams, of a reactant at time t seconds is given by $M = G(t) = \frac{18}{t+1}$.

在一化學反應中，一反應物在時間 t **秒時的質量** M (**單位：**g) **為：** $M = G(t) = \frac{18}{t+1}$。

(a) Find the average rate of change of the mass from $t = 3$ to $t = 4$ seconds.
試求該反應物的質量從 $t = 3$ **至** $t = 4$ **秒的平均變化率。** (3 **分**)

(b) Find the average rate of change of the mass from $t = 3$ to $t = 3.2$ seconds.
試求該反應物的質量從 $t = 3$ **至** $t = 3.2$ **秒的平均變化率。** (2 **分**)

(c) (i) Find **試求** $\lim_{\Delta t\to 0}\frac{G(3+\Delta t)-G(3)}{\Delta t}$。

(ii) Interpret your answer in **(i)**. **試解釋** (i) **的結果。** (5 **分**)

B3. Let **設** $x = t+\frac{1}{t}$ and **及** $y = t-\frac{1}{t}$。

(a) Find $\frac{dy}{dx}$ at $t = 2$ by parametric differentiation.

利用參數的微分法，求 $\frac{dy}{dx}$ **在** $t = 2$ **時的值。** (5 **分**)

(b) (i) Show that **證明** $x^2 - y^2 = 4$。

(ii) Hence, find $\frac{dy}{dx}$ at $t = 2$ by implicit differentiation.

由此，利用隱函數的微分法，求 $\frac{dy}{dx}$ **在** $t = 2$ **時的值。** (7 **分**)

B4. (a) If x is a function of t and $\frac{dx}{dt} \neq 0$, show that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

若 x 為 t 的函數，且 $\frac{dx}{dt} \neq 0$，證明 $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$。 (3 分)

(b) If $\frac{dy}{dx} = f(t)$, where $f(t)$ is a function of t, show that $\frac{d^2y}{dx^2} = \frac{\frac{df(t)}{dt}}{\frac{dx}{dt}}$.

若 $\frac{dy}{dx} = f(t)$，其中 $f(t)$ 為 t 的函數，證明 $\frac{d^2y}{dx^2} = \frac{\frac{df(t)}{dt}}{\frac{dx}{dt}}$。 (3 分)

(c) Given the parametric equations $x = \cos^3\theta - 3\cos\theta$ and $y = 3\sin\theta - \sin^3\theta$, find $\frac{d^2y}{dx^2}$.

已知參數方程為 $x = \cos^3\theta - 3\cos\theta$ 及 $y = 3\sin\theta - \sin^3\theta$，試求 $\frac{d^2y}{dx^2}$ 的值。 (10 分)

B5. (a) Show that for any constants a and b, $y = ax^2 + bx$ satisfies the equation
證明對於任意常數 a 及 b，$y = ax^2 + bx$ 滿足方程

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \text{。}$$

(b) Find the values of a and b so that y also satisfies the condition that when $x = 1$, $y = 3$ and $\frac{dy}{dx} = 8$.

試求 a 及 b 的值，使得當 $x = 1$，$y = 3$ 及 $\frac{dy}{dx} = 8$ 時，(a) 中方程仍然成立。 (4 分)

B6. Let 設 $y = \sqrt{a^2 - x^2}$，where a is a constant.其中 a 為一常數。

(a) Find 試求 $\frac{dy}{dx}$ 的值。 (3 分)

(b) Find 試求 $\frac{d^2y}{dx^2}$ 的值。 (3 分)

(c) Hence 由此，show that 證明 $x(a^2 - x^2)\frac{d^2y}{dx^2} - a^2\frac{dy}{dx} = 0$。 (3 分)

B7. The normal to the curve $y = (x - 2)^2$ at the point A(3, 1) meets the curve again at B. Find
曲線 $y = (x - 2)^2$ 在點 A(3, 1) 的法線與之在點 B 相交。試求

(a) the equation of the normal to the curve at A,
曲線在 A 的法線方程, (4 分)

(b) the coordinates of B,
B 的坐標, (3 分)

(c) the angle between the tangent to the curve at B and the chord AB.
曲線在 B 的切線與弦 AB 之間的夾角。 (5 分)

B8. The equation of a curve is $4y^2 = x^3$.
一曲線的方程為 $4y^2 = x^3$。

(a) Show that the point P$(4t^2, 4t^3)$ is a point on the curve.
證明點 P$(4t^2, 4t^3)$ 在曲線上。 (2 分)

(b) Find the equation of the tangent to the curve at the point P.
求曲線在 P 的切線方程。 (3 分)

(c) If the tangent in **(b)** meets the curve again at Q, find the coordinates of Q.
若 (b) 中的切線與曲線在 Q 相交，求 Q 的坐標。 (3 分)

(d) If M is the midpoint of PQ, find the equation of the locus of Q.
若 M 為 PQ 的中點，求 M 的軌跡方程。 (4 分)

B9. The horizontal displacement x and vertical displacement y of a particle from the origin O at time t are given by $x = t^3 - 7t^2 + 8t$ and $y = \sqrt{2t+1}$.

一質點於時間 t，由原點 O 的水平位移 x 和垂直位移 y 定義為 $x = t^3 - 7t^2 + 8t$ 及 $y = \sqrt{2t+1}$。

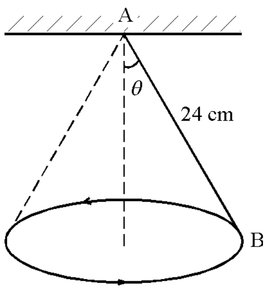
(a) Find the horizontal and vertical velocities of the particle at time $t = 3$.
求質點在時間 $t = 3$ 時的水平速度和垂直速度。 (3 分)

(b) Hence, find the velocity of the particle at $t = 3$.
由此，求質點在 $t = 3$ 時的速度。 (3 分)

(c) Find the time interval at which the horizontal velocity of the particle is negative.
求質點的水平速度為負值時的時間區間。 (3 分)

(d) Find the vertical acceleration of the particle at $t = 4$.
求質點在 $t = 4$ 時的垂直加速度。 (3 分)

B10.



A rod AB 24 cm long is hinged with the end A on a ceiling. It is rotated about the vertical line through A to generate a cone. Let θ be the semi-vertical angle of the cone and V cm^3 be the volume of the cone.
一支長 24 cm 的棍棒 AB，其一端 A 懸於天花板。該棍棒繞著經過 A 的垂直線旋轉而形成一圓錐體。設圓錐體的半頂角為 θ 及體積為 V cm^3。

(a) Express V in terms of θ.
試以 θ 表 V 。 (3 分)

(b) If θ increases at the rate of $\frac{\pi}{30}$ radians per second, find
若 θ 的遞增率為 $\frac{\pi}{30}$ 弧度/s，求

(i) the rate of change of V when $\theta = \frac{\pi}{6}$.
當 $\theta = \frac{\pi}{6}$ 時，V 的變化率，

(ii) the range of θ such that V is increasing.
θ 的範圍使 V 遞增。 (7 分)

B11. Let **設** $f(x) = \frac{x^2 + 9}{2x}$。

(a) Find **求** $f'(x)$ and **及** $f''(x)$。 (3 分)

(b) Find the range of values of x such that $f(x)$ is increasing.
求 x 值的範圍使 $f(x)$ 遞增。 (3 分)

(c) Find the maximum and minimum values of x.
求 $f(x)$ 的極大值及極小值。 (6 分)

B12. A function is given by $y = 4 \sin x + k \cos 2x$ for $0 \le x \le \pi$ and k is a positive constant. It has a stationary point at $x = \frac{\pi}{6}$.

一函數定義為 $y = 4 \sin x + k \cos 2x$ $(0 \le x \le \pi)$，而 k 是一正常數。已知函數在 $x = \frac{\pi}{6}$ 有一駐點。

(a) Find the value of k.
求 k 的值。 (5 分)

(b) Find the turning points of the function.
求函數的轉向點。 (5 分)

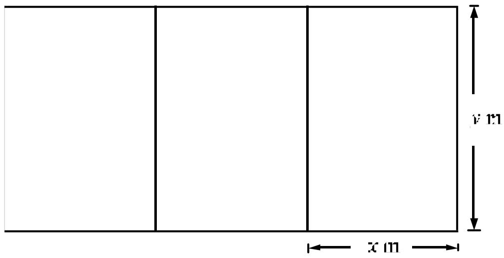
B13. Let **設函數** $f(x) = \frac{x}{x^2 + 4}$。

(a) Is $f(x)$ an odd function or an even function?
$f(x)$ 是奇函數還是偶函數？ (2 分)

(b) Find the turning points of the function.
求此函數的轉向點。 (6 分)

(c) Sketch the graph of $y = f(x)$.
描繪 $y = f(x)$ 的圖像。 (4 分)

B14.



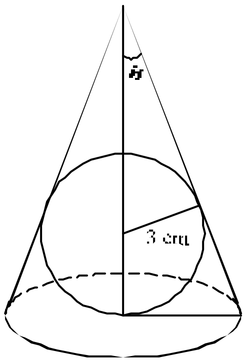
A zoologist is conducting a controlled experiment on breeding sheep. He/She constructs 3 identical rectangular pens using 1200 m of fencing as shown in the figure.
一動物學家正進行一項綿羊繁殖的受控實驗。他利用 1200 m **的籬笆建造三個相同的長方形圍欄，如圖所示。**

(a) Express y in terms of x.
試以 x **表** y。 (2 **分**)

(b) Express the total area A, in m^2, in terms of x.
試以 x **表總面積** A (**單位：** m^2) 。 (2 **分**)

(c) Find the maximum area he/she can enclose.
求他能圍繞的最大面積。 (6 **分**)

B15.



A right circular cone is circumscribed to a sphere of radius 3 cm, with the base of the cone touching the sphere. Let θ be the semi-vertical angle of the cone and V cm^3 be its volume.
一半徑為 3 cm **的球體內接於一正圓錐體。設圓錐體的半頂角為** θ **及體積為** V cm^3。

(a) Express V in terms of θ.
試以 θ **表** V。 (3 **分**)

(b) Find the range of values of θ such that V is decreasing.
求 θ **的範圍使** V **遞減。** (4 **分**)

(c) Hence, find the minimum volume of the cone.
由此，求圓錐體的最小體積。 (5 **分**)

B16. (a) Show that the volume of a right circular cone of slant height 6 cm and semi-vertical angle θ is given by $V = 72\pi\sin^2\theta\cos\theta$.
證明斜邊長 6 cm **及半頂角為** θ **的正圓錐體的體積為** $V = 72\pi\sin^2\theta\cos\theta$。 (4 **分**)

(b) Find the value of θ for which V is a maximum. Hence, find the largest volume.
求該圓錐體取最大體積時 θ **的值。由此，求該最大體積。** (6 **分**)

B17. (a) The function $f(x) = x^3 + ax^2 + bx + c$ has a maximum at $x = -1$ and a minimum value -6 at $x = 3$. Find the values of the constants a, b and c.
函數 $f(x) = x^3 + ax^2 + bx + c$ **在** $x = -1$ **有最大值而在** $x = 3$ **的最小值為** -6。**求常數** a、b **及** c **的值。**
(6 **分**)

(b) Sketch the graph of $y = f(x)$ for $-2 \le x \le 4$.
描繪 $y = f(x)$ **在** $-2 \le x \le 4$ **的圖像。** (4 **分**)

B18. (a) Sketch the curves $y = 6x - x^2$ and $y = 2x^2 - 6x$.
描繪曲線 $y = 6x - x^2$ **及** $y = 2x^2 - 6x$ **的圖像。** (4 **分**)

(b) Find the angles at which the curves intersect.
求曲線相交的角度。 (8 **分**)

完