

Fast Marching farthest point sampling for implicit surfaces and point clouds

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Abstract

In a recent paper [13], the Fast Marching farthest point sampling strategy (FastFPS) for planar domains and curved manifolds was introduced. The version of FastFPS for curved manifolds discussed in the paper [13] deals with surface domains in triangulated form only. Due to a restriction of the underlying Fast Marching method, the algorithm further requires the splitting of any obtuse into acute triangles to ensure the consistency of the Fast Marching approximation. In this paper, we overcome these restrictions by using Mévoli and Sapiro's [11, 12] extension of the Fast Marching method to the handling of implicit surfaces and point clouds. We find that the extended FastFPS algorithm can be applied to surfaces in implicit or point cloud form without the loss of the original algorithm's computational optimality and without the need for any preprocessing.

1 Introduction

In Moenning and Dodgson [13], the notion of Fast Marching farthest point sampling (FastFPS) for planar domains and triangulated curved manifolds is put forward. FastFPS makes use of Fast Marching [22, 23] for the incremental construction of uniform or non-uniform distance maps across the sampling domain. As a result, an efficient progressive sampling method based on the uniform farthest point principle introduced by Eldar et al. [5, 6] and sharing its favourable properties such as excellent anti-aliasing properties, a high data acquisition rate and an elegant relationship to the Voronoi diagram concept is obtained. In addition, FastFPS provides a natural extension of the farthest point principle to the case of non-uniform, adaptive progressive sampling without the loss of computational optimality. A sampling technique featuring these properties is of particular interest for applications such as uniform and feature-sensitive point cloud simplification with density guarantee, progressive transmission of image or 3D surface data [10], implicit surface sampling and rendering [24], (progressive) rendering [14], progressive acquisition [13] and machine vision [25].

FastFPS for triangulated surfaces requires the splitting of any obtuse into acute triangles in a preprocessing step [13]. Although this preprocessing step does not add significantly to the algorithm’s computational complexity, it affects its accuracy [7]. Furthermore, numerical analysis over polygonal surfaces is generally less accurate and robust than numerical analysis over Cartesian grids [4]. Finally, the surface domain may not be readily available in triangulated form or the computation of a triangulation may involve a performance penalty since it may otherwise not be required by the application.

Using Mémoli and Sapiro’s [11, 12] recent augmentation of the Fast Marching concept, we propose an extended FastFPS method which may be applied to surfaces in implicit or point cloud form without the need for any preprocessing or prior triangulation. Surfaces given in triangulated form may be dealt with by either using FastFPS for triangulated surfaces [13] or implicitising the surface in a preprocessing step. The method works on a Cartesian grid and retains the efficiency of FastFPS for triangulated surfaces [13].

We briefly review both the Voronoi diagram and the farthest point sampling as well as the “conventional” Fast Marching concept and its extension suggested by Mémoli and Sapiro [11, 12]. We then introduce our FastFPS algorithm for implicit surfaces and point clouds and present a worked example for its application to point clouds. We conclude with a brief summary and discussion.

2 Previous Work

2.1 Voronoi diagrams

Given a finite number n of distinct data sites $P := \{p_1, p_2, \dots, p_n\}$ in the plane, for $p_i, p_j \in P, p_i \neq p_j$, let

$$B(p_i, p_j) = \{t \in \mathbb{R}^2 \mid d(p_i - t) = d(p_j - t)\} \quad (1)$$

where d may be an arbitrary distance metric provided the bisectors with regard to d remain curves bisecting the plane. $B(p_i, p_j)$ is the perpendicular bisector of the line segment $\overline{p_i p_j}$. Let $h(p_i, p_j)$ represent the half-plane containing p_i bounded by $B(p_i, p_j)$. The Voronoi cell of p_i with respect to point set $P, V(p_i, P)$, is given by

$$V(p_i, P) = \bigcap_{p_j \in P, p_j \neq p_i} h(p_i, p_j) \quad (2)$$

That is, the Voronoi cell of p_i with respect to P is given by the intersection of the half-planes of p_i with respect to $p_j, p_j \in P, p_j \neq p_i$.

If p_i represents an element on the convex hull of $P, V(p_i, P)$ is unbounded. For a finite domain, the bounded Voronoi cell, $BV(p_i, P)$, is defined as the conjunction of the cell $V(p_i, P)$ with the domain.

The boundary shared by a pair of Voronoi cells is called a Voronoi edge. Voronoi edges meet at Voronoi vertices.

The Voronoi diagram of P is given by

$$VD(P) = \bigcup_{p_i \in P} V(p_i, P) \quad (3)$$

The bounded Voronoi diagram, $BVD(P)$, follows correspondingly as:

$$BVD(P) = \bigcup_{p_i \in P} BV(p_i, P) \quad (4)$$

Figure 1(a) shows an example of a bounded Voronoi diagram. Note that the Voronoi diagram concept extends to higher dimensions. For more detail, see the comprehensive treatment by Okabe et al. [15] or the survey article by Aurenhammer [2].

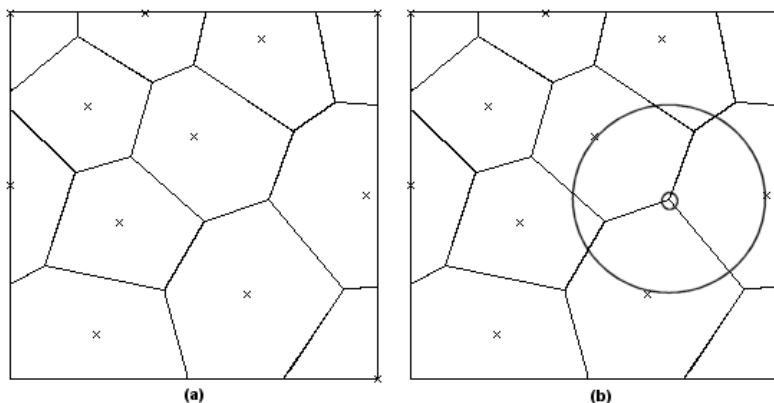


Figure 1: (a) Bounded Voronoi diagram of 12 sites in the plane. (b) The next farthest point sample (sample point 13) is located at the centre of the largest circle empty of any other sample site.

2.2 Farthest point sampling

Farthest point sampling is based on the idea of repeatedly placing the next sample point in the middle of the least-known area of the sampling domain. In the following, we summarise the reasoning underlying this approach for both the uniform and non-uniform case presented in Eldar et al. [5, 6].

Starting with the uniform case, Eldar et al. [5, 6] consider the case of an image representing a continuous stochastic process featuring constant first and second order central moments with the third central moment, i.e., the covariance, decreasing (exponentially) with spatial distance. That is, given a pair of sample points $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$, the points' correlation, $E(p_i, p_j)$, is assumed to decrease with the Euclidean distance, d_{ij} , between the points

$$E(p_i, p_j) = \sigma^2 e^{-\lambda d_{ij}} \quad (5)$$

with $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

Based on their linear estimator, the authors subsequently put forward the following representation for the mean square error, i.e., the deviation from the “ideal” image resulting from estimation error, after the N th sample

$$\varepsilon^2(p_0, \dots, p_{N-1}) = \iint \sigma^2 - U^T R^{-1} U \, dx \, dy \quad (6)$$

where

$$R_{ij} = \sigma^2 e^{-\lambda \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}$$

and

$$U_i = \sigma^2 e^{-\lambda \sqrt{(x_i - x)^2 + (y_i - y)^2}}$$

for all $0 \leq i, j \leq N - 1$. The assumption of stationary first and second order central moments has therefore yielded the result that the expected mean square (reconstruction) error depends on the location of the $N + 1$ th sample only. Since stationarity implies that the image’s statistical properties are spatially invariant and given that point correlations decrease with distance, uniformly choosing the $N + 1$ th sample point to be that point which is farthest away from the current set of sample points therefore represents the optimal sampling approach within this framework.

This sampling approach is intimately linked with the incremental construction of a Voronoi diagram over the image domain. To see this, note that the point farthest away from the current set of sample sites, S , is represented by the centre of the largest circle empty of any site $s_i \in S$. Eldar et al. [6] show that in the case of a farthest point sequence, the centre of such a circle is given by a vertex of the bounded Voronoi diagram of S , $BVD(S)$. Thus, as indicated in figure 1(b), incremental (bounded) Voronoi diagram construction provides sample points progressively.

From visual inspection of images it is clear that usually not only the sample covariances but also the sample means and variances vary spatially across an image. When allowing for this more general variability and thus turning to the design of a non-uniform, adaptive sampling strategy, the assumption of sample point covariances decreasing, exponentially or otherwise, with point distance remains valid. However, since Voronoi diagrams in non-uniform metrics may lose favourable properties such as cell connectedness, Eldar et al. [5, 6] opt for the non-optimal choice of augmenting their model by an application-dependent weighting scheme for the vertices in the Euclidean Voronoi diagram.

2.3 Fast Marching

Fast Marching represents a very efficient technique for the solution of front propagation problems which can be formulated as boundary value partial differential equations. We show that the problem of computing the distance map across a smooth sampling domain can be posed in the form of such a partial differential equation and outline the Fast Marching approach towards approximating its solution.

For simplicity, take the case of an interface propagating with speed function $F(x, y, z)$ away from a source (boundary) point (u, v, w) across a 3D Euclidean domain. When interested in the time of arrival, $T(x, y, z)$, of the interface at grid point (x, y, z) , i.e., the distance map T given source point (u, v, w) , the relationship between the magnitude of

the distance map's gradient and the given weight $F(x, y, z)$ at each point can be expressed as the following boundary value formulation

$$|\nabla T(x, y, z)| = F(x, y, z) \quad (7)$$

with boundary condition $T(u, v, w) = 0$.

That is, the distance map gradient is proportional to the weight function. The problem of determining a weighted distance map has therefore been transformed into the problem of solving a particular type of Hamilton-Jacobi partial differential equation, the Eikonal equation [8]. For $F(x, y, z) > 0$, this type of equation can be solved for $T(x, y, z)$ using Fast Marching.

Since the Eikonal equation is well-known to become non-differentiable through the development of corners and cusps during propagation, the Fast Marching method considers only upwind, entropy-satisfying finite difference approximations to the equation thereby consistently producing weak solutions. As an example for a first order approximation to the gradient operator, consider [17]

$$\begin{aligned} & [\max(D_{ijk}^{-x}T, -D_{ijk}^{+x}T, 0)^2 + \\ & \max(D_{ijk}^{-y}T, -D_{ijk}^{+y}T, 0)^2 + \\ & \max(D_{ijk}^{-z}T, -D_{ijk}^{+z}T, 0)^2]^{1/2} = F_{ijk} \end{aligned} \quad (8)$$

where $F_{ijk} \equiv F(i\Delta x, j\Delta y, k\Delta z)$. $D_{ijk}^{-x}T \equiv \frac{T_{ijk} - T_{i-1jk}}{h}$ and $D_{ijk}^{+x}T \equiv \frac{T_{i+1jk} - T_{ijk}}{h}$ are the standard backward and forward derivative approximation with h representing the grid spacing; equivalently for $D_{ijk}^{-y}T$, $D_{ijk}^{+y}T$, $D_{ijk}^{-z}T$ and $D_{ijk}^{+z}T$. T_{ijk} is the discrete approximation to $T(i\Delta x, j\Delta y, k\Delta z)$ on a regular Cartesian grid.

This upwind difference approximation implies that information propagates from smaller to larger values of T only, i.e., a grid point's arrival time gets updated by neighbouring points with smaller T values only. This monotonicity property allows for the maintenance of a narrow band of candidate points around the front representing its outward motion. The property can further be exploited for the design of a simple and efficient algorithm by freezing the T values of existing points and subsequently inserting neighbouring ones into the narrow band thereby marching the band forward. The basic Fast Marching algorithm can thus be summarised as follows [22, 23]

- 0) Mark an initial set of grid points as ALIVE. Mark as CLOSE, all points neighbouring ALIVE points. Mark all other grid points as FAR.
- 1) Let TRIAL denote the point in CLOSE featuring the smallest arrival time. Remove TRIAL from CLOSE and insert it in ALIVE.
- 2) Mark all neighbours of TRIAL which are not ALIVE as CLOSE. If applicable, remove the neighbour under consideration from FAR.
- 3) Using the gradient approximation, update the T values of all neighbours of TRIAL using only ALIVE points in the computation.
- 4) Loop from 1).

Arrangement of the elements in CLOSE in a min-heap [20] leads to an $O(N \log N)$ implementation, with N representing the number of grid points. Note that a single min-heap structure may be used to track multiple propagation fronts originating from different points in the domain.

Unlike other front propagation algorithms [3], each grid point is only touched once, namely when it is assigned its final value. Furthermore, the distance map $T(x, y, z)$ is computed with “sub-pixel” accuracy, the degree of which varies with the order of the approximation scheme and the grid resolution. In addition, the distance map is computed directly across the domain, a separate binary image indicating the source points is not required. Finally, since the arrival time information of a grid point is only propagated in the direction of increasing distance, the size of the narrow band remains small. Therefore, the algorithm’s complexity is closer to the theoretical optimum of $O(N)$ than $O(N \log N)$ [22].

Kimmel and Sethian [7, 8] extend this “conventional” Fast Marching framework to triangulated surfaces. Their algorithm, however, requires the splitting of any obtuse into acute triangles as part of a preprocessing step. In the case of surfaces given in implicit or point cloud form, this preprocessing step would further involve the triangulation of the domain which may otherwise not be needed by the application.

Mémoli and Sapiro [11, 12] put forward an extension of the “conventional” Fast Marching method which allows for the computation of distance functions on implicit surfaces or point clouds in three or higher dimensions without the need for any prior triangulation of the domain.

Let M represent a closed hyper-surface in \mathbb{R}^m given as the zero level-set of a distance function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$. The r -offset, Ω_r , of M is given by the union of the balls centred at the surface points with radius r

$$\Omega_r := \bigcup_{x \in M} B(x, r) = \{x \in \mathbb{R}^m : |\phi(x)| \leq r\} \quad (9)$$

For a smooth M and r sufficiently small, Ω_r is a manifold with smooth boundary [11]. To compute the weighted distance map originating from a source point $q \in M$ on M , Mémoli and Sapiro [11] suggest using the Euclidean distance map in Ω_r to approximate the intrinsic distance map on M . That is

$$|\nabla_M T_M(p)| = F(p) \quad (10)$$

for $p \in M$ and with boundary condition $T_M(q) = 0$ is approximated by

$$|\nabla T_{\Omega_r}(p)| = \tilde{F}(p) \quad (11)$$

for $p \in \Omega_r$ and boundary condition $T_{\Omega_r}(q) = 0$. \tilde{F} represents the (smooth) extension of F on M into Ω_r .

The problem of computing an intrinsic distance map has therefore been transformed into the problem of computing an extrinsic distance map in an Euclidean manifold with boundary. Mémoli and Sapiro [11, 12] show that the approximation error between these two distance maps is of the same theoretical order as that of the Fast Marching algorithm. With the order of the numerical approximation remaining unchanged, the Fast Marching method can be used to approximate the solution to (11) in a computationally optimal manner by only slightly modifying the Fast Marching technique to deal with bounded spaces as follows

- 0) Mark an initial set of grid points in Ω_r as ALIVE. Mark as CLOSE, all points which neighbour ALIVE points and which fall Ω_r . Mark all other grid points in Ω_r as FAR.
- 1) Let TRIAL denote the point in CLOSE featuring the smallest arrival time. Remove TRIAL from CLOSE and insert it in ALIVE.
- 2) Mark all neighbours of TRIAL which belong to Ω_r and which are not ALIVE as CLOSE. If applicable, remove the neighbour under consideration from FAR.
- 3) Using the gradient approximation, update the T values of all neighbours of TRIAL in CLOSE using only ALIVE points in the computation.
- 4) Loop from 1).

3 Fast farthest point sampling for implicit surfaces and point clouds

In the following, we present the extension of the original FastFPS algorithm [13] to implicit surfaces and point clouds. For simplicity, we consider the case of sampling from surfaces in 3D and start with the uniform case.

The algorithm proceeds with the embedding of the given implicit surface or point cloud in a Cartesian grid sufficiently large as to allow for a thin offset band, Ω_r , around the surface. To include the stencil used in a finite difference approximation such as (8), the radius r needs to be at least as large as $\Delta x \sqrt{m}$, where Δx denotes the grid size. The radius is bounded from above by the minimum of the maximal radius which does not cause an intersecting boundary or an unconnected domain and the inverse of a bound for the absolute sectional curvature of M . These bounds have to be met by r to be able to obtain both a smooth extension of F on M into Ω_r and a smooth boundary for Ω_r . Note that alongside the grid size Δx , the radius r may be allowed to vary locally [11].

Given an initial set of sample points S in Ω_r , we construct $BVD(S)$ by simultaneously propagating fronts from each of the initial sample points outwards. During this propagation, only points located in Ω_r are considered. This process is equivalent to the computation of the Euclidean distance map across the domain given S and Ω_r . It is achieved by solving the Eikonal equation (7) with $F(x, y, z) = 1$ and using a single min-heap.

The vertices of $BVD(S)$ are given by those grid points entered by three or more propagation waves (or two for points on the domain boundary) and are therefore obtained as a by-product of the propagation process. The Voronoi vertices' arrival times are inserted into a max-heap data structure. The algorithm then proceeds by extracting the root from the max-heap, the grid location of which represents the location of the next farthest point sample. The sample is inserted into $BVD(S)$ by resetting its arrival time to zero and propagating a front away from it. The front will continue propagating until it hits grid points featuring lower arrival times and thus belonging to a neighbouring Voronoi cell. The T values of updated grid points are updated correspondingly in the max-heap using back pointers. New and obsolete Voronoi vertices are inserted or removed from the max-heap respectively. The algorithm continues extracting the root from the max-heap until

it is empty or the sample point budget has been exhausted. Since points are sampled in Ω_r , the equivalent sample points on the implicit surface or point cloud are found at any given time by projecting the sample points in Ω_r onto the surface.

By allowing $F(x, y, z)$ to vary with any (positive) weights associated with points in the domain, this algorithm is easily extended to the case of non-uniform, adaptive sampling. The algorithm can thus be summarised as follows

- 0) Embed the given surface in a Cartesian grid sufficiently large to allow for an offset band of size r around the surface. Given an initial sample set $S \in \Omega_r$, $n = |S| \geq 1$, compute $BVD(S)$ by propagating fronts with speed F_{ijk} from the sample points outwards using “extended” Fast Marching. Store the Voronoi vertices’ arrival times in a max-heap.
- 1) Extract the root from the max-heap to obtain s_{n+1} . $S' = S \cup \{s_{n+1}\}$. Compute $BVD(S')$ by propagating a front locally from s_{n+1} outwards using Fast Marching and a finite difference approximation such as (8).
- 2) Correct the arrival times of updated grid points in the max-heap. Insert the vertices of $BV(s_{n+1}, S')$ in the max-heap. Remove obsolete Voronoi vertices of the neighbours of $BV(s_{n+1}, S')$ from the max-heap.
- 3) If neither the max-heap is empty nor the point budget has been exhausted, loop from 1).

Extracting the root from, inserting into and removing from the max-heap with subsequent re-heapifying are $O(\log W)$ operations, where W represents the number of elements in the heap. W is $O(N)$, N representing the number of grid points. The updating of existing max-heap entries is $O(1)$ due to the use of back pointers from the grid to the heap. The detection of a (bounded) Voronoi cell’s vertices and boundary is a by-product of the $O(N \log N)$ front propagation. Thus, the algorithm’s asymptotic efficiency is $O(N \log N)$.

4 Worked example

We consider the problem of coarse-to-fine simplification of a dense point cloud, $P = \{p_1, p_2, \dots, p_{N_1}\}$, to a target model size $N_2 < N_1$. The point set is assumed to represent the surface of a smooth, two-manifold boundary of a 3D model (here: “Venus”). The original point cloud’s oversampling is a typical data acquisition result and makes the reduction of its inherent redundancy particularly useful before any kind of further processing such as surface reconstruction is attempted. For the purpose of coarse-to-fine simplification, we sample the point cloud using FastFPS for point clouds within seconds and render the polygonal meshes reconstructed from the uniform point sets produced by our algorithm.

To apply FastFPS for point clouds to this point set, we embed the surface representation in a 3D Cartesian grid. The FastFPS algorithm is then used to sample the point cloud irregularly uniformly following the automatic selection of a starting sample point within the outside offset band surrounding the surface. Note that for simplicity, r was set to a uniform $2\Delta x$.

Figure 2 presents the simplified point sets and the renderings of the reconstructed polygonal meshes for target model sizes of 97.5%, 95.0% and 90.0% of the size of the

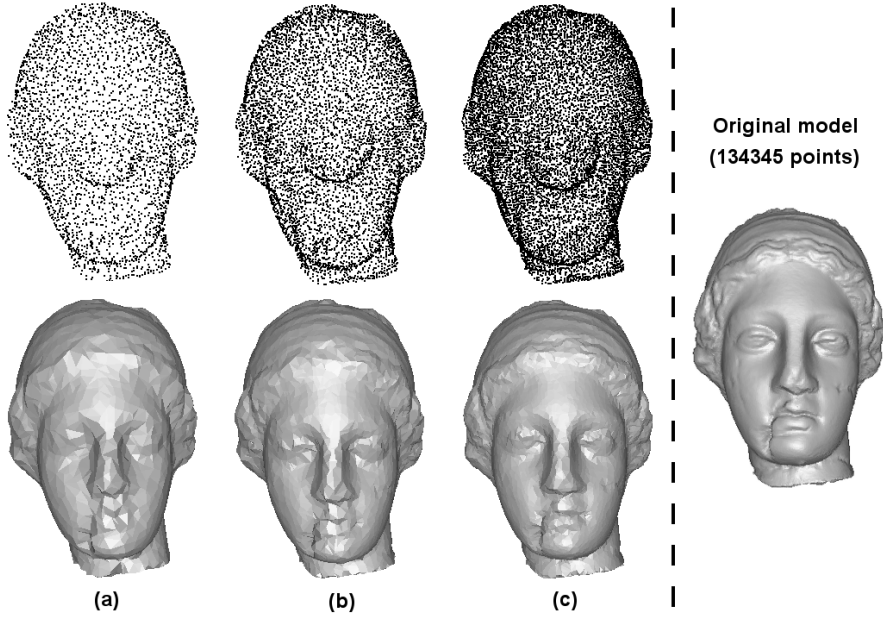


Figure 2: Uniformly simplified point sets produced by FastFPS for point clouds for target model sizes of (a) 97.5% (b) 95.0% (c) 90.0% of the input model size.

original point set respectively. Qualitatively, the irregularly uniformly simplified point sets yield a cluster- and hole-free coverage of the surface and allow for visually appealing reconstructions even for relatively strongly simplified target models. A more objective, quantitative analysis of the approximation error is beyond the scope of this report and will be reported elsewhere.

5 Conclusions

We presented an extension of the Fast Marching farthest point sampling principle introduced in Moenning and Dodgson [13] to the case of surfaces given in implicit or point cloud form. By making use of a recently proposed extension to the Fast Marching framework [11, 12], this extended FastFPS algorithm retains the computational efficiency and ease of implementation of the basic FastFPS algorithm. The restrictive requirement of (acutely) triangulated domains associated with the basic FastFPS algorithm for surfaces [13] is therefore overcome by the suggested extension and a new implicit surface or point cloud sampling algorithm yielding uniform or adaptive sample point sets is obtained. Possible applications are numerous and include uniform and feature-sensitive point cloud simplification, implicit surface sampling and rendering, progressive acquisition, surface encoding and compression, progressive transmission and progressive rendering.

As regards further FastFPS-related research, we are currently working on detailed comparative studies regarding the quality of FastFPS sample point sets generated from images and surfaces relative to sample point sets produced by relevant alternative sampling strategies. This work includes the analysis of any explicit guarantees which can be made regarding the nature of FastFPS sample point sets. On the application side, we are interested in using uniform FastFPS surface sample point sets alongside subdi-

vision displacement maps [9] for multiresolution surface representations. In the case of point clouds, point-based (multiresolution) representations for point-based rendering are readily available [1, 10, 16]. With regard to adaptive point cloud sampling, we are investigating point-based curvature approximations using eigenanalysis of local point set neighbourhoods. Finally, we are exploring the use of FastFPS as a link between the data acquisition and surface reconstruction/rendering steps in a fully integrated and automated surface processing pipeline [18, 19]. With the help of point-based curvature approximations and FastFPS, previously collected samples may be used for the determination of future sample point locations thereby, for example, concentrating relatively more samples in regions of high curvature.

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