Chapter Seven: Basic Probability

When you completed this chapter, you will be able to:

- ✓ understand of the basic concepts of probability;
- ✓ understand the meanings and definitions of the probability terminology;
- ✓ identify/prove the relationship between events;
- ✓ identify the basic approaches of probability;
- ✓ apply the counting rules into probability concepts;
- ✓ distinguish combination from permutation;
- ✓ understand and recognise the inter-relationship between probability rules; and
- ✓ apply the probability rules in real case problems.

Reference(s): Mason Chapter 5, Berenson Chapter 6, Freund Chapter 4 and 5
Exercise(s): Seminars 12, 13, 14 and 15, Mason Chapter 5 Exercises 5, 7-11, 19-22, 27, 32-34, 40, 43-45, and 47-52.

Probability is the likelihood or chance that a particular event will occur. It could refer to:

- the chance of picking a black card from a deck of cards.
- the chance that a house selected at random from the real estate survey has a modern kitchen.
- the chance that a new consumer product on the market will be successful.

Basic Probability Terminology

Experiment: An experiment is a ***planned process*** of measurement or observation of ***different outcomes***, where the outcomes cannot be predicted with certainly.

e.g.1 Tossing a 6-sided die and observing the number showing face up when the die comes to rest.

Sample Point: Sample point is a ***single*** *outcome* of an experiment.

e.g.2 The number facing up may be 3, then 3 is the sample point of the experiment.

Sample Space: Sample Space (S) is the collection of ***all possible*** *outcomes* of an experiment.

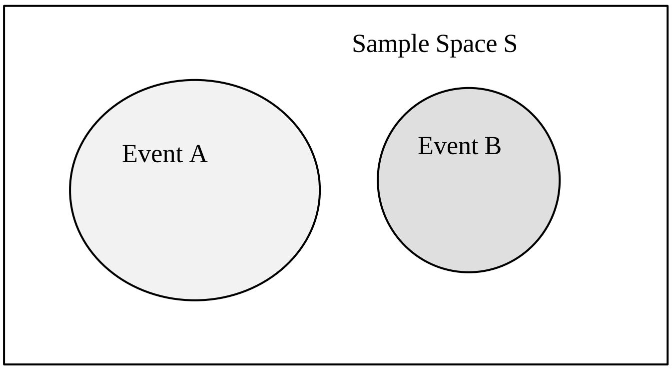
e.g.3 All possible outcomes for tossing a 6-sided die are {1, 2, 3, 4, 5, and 6}, so it is the sample space of the experiment.

Event: An event is any ***subset*** *of a sample space*.

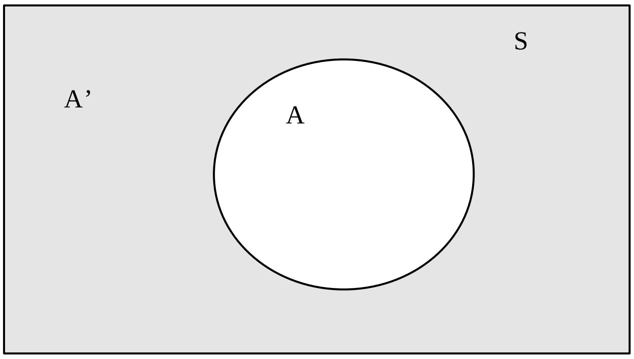
e.g.4 Event X is defined as all even number outcomes, and
Event Y is defined as number 1 and 3.

Venn diagram: Venn diagram is a ***graphical*** representation of a sample space and events.

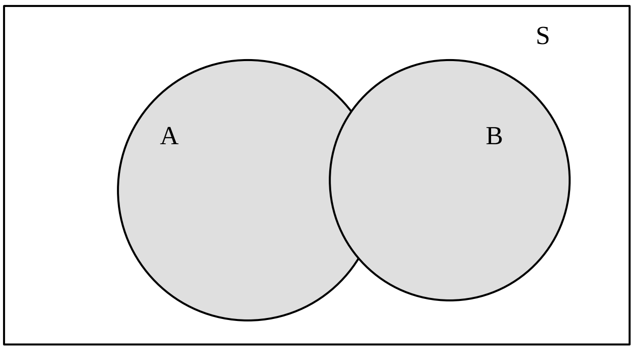
e.g.5



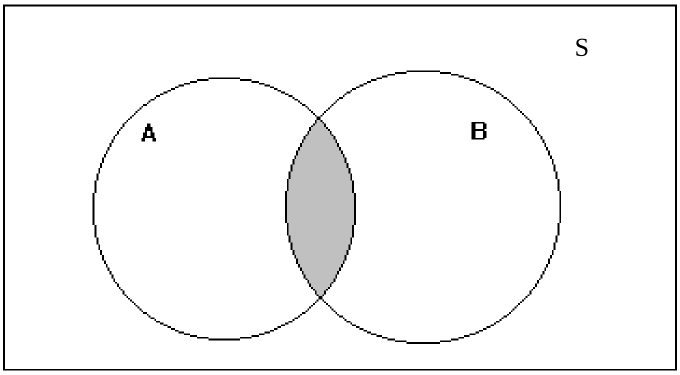
Complement of an Event: If A is an event contained in a sample space S, then the event not A, denoted by A' or $\overline{A}$, is the event containing all the outcomes in S that are *not contained* in A.



Union of Events: is the event that A will occur ***or*** B will occur ***or*** they both occur.
(A È B)



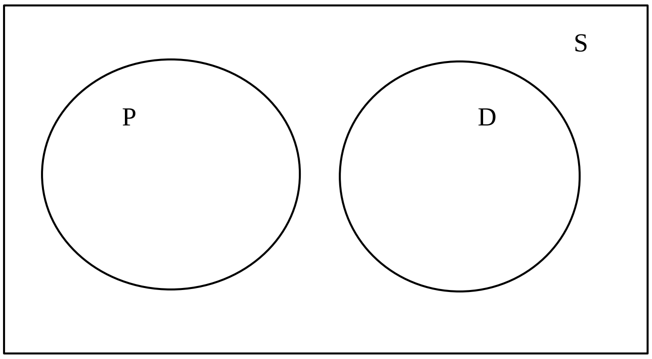
Intersection of Events: is the event that both A ***and*** B will occur *at the same time.*
(A Ç B)



Mutually Exclusive Events: If A and B are events that have ***no outcome in common***, then A and B are called *mutually exclusive* events, that is
(A Ç B) = Æ

e.g.6 Three men, Peter, David and John, are standing for election as chairman of a committee. Let P be the event "Peter is elected" and D be the event "David is elected".

The P and D are mutually exclusive events as both cannot be elected as chairman.

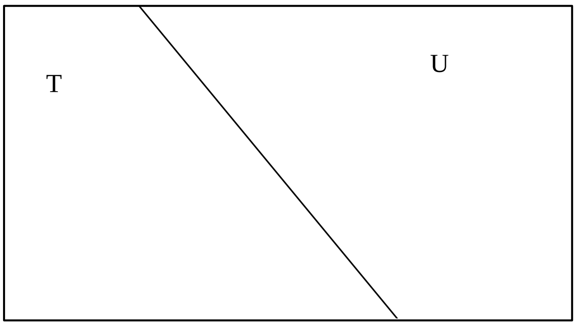


Mutually Exclusive Events P and D

Collectively Exhaustive Events: If two events A and B are such that $(A \cup B) = S$, then the events A and B are said to be *collectively exhaustive.*

In the last example, P and D are mutually exclusive events, but not collectively exhaustive, since P and D together do not form the whole sample space, and John may be selected.

e.g.7 Sample Space S



Event T and U are mutually exclusive and collectively exhaustive since T and U together form the whole sample space. In other words, *either T or U must occur*.

Types of Probability Approach

1. Classical Approach

It is based on an assumption that the outcomes of an experiment are ***equally likely***.

$$\textbf{Probability of an event} = \frac{\textbf{Number of outcomes favourable to occurence of the event}}{\textbf{Total number of possible outcomes}}$$

e.g.8 Consider the experiment of rolling a six-sided die? What is the probability of the event “an even number of spot appear face up”?

There are six possible outcomes:
{one-spot, two-spot, three-spot, four-spot, five-spot, and six-spot}

And there are three “Favourable” outcomes, i.e. two-spot, four-spot, and six-spot.

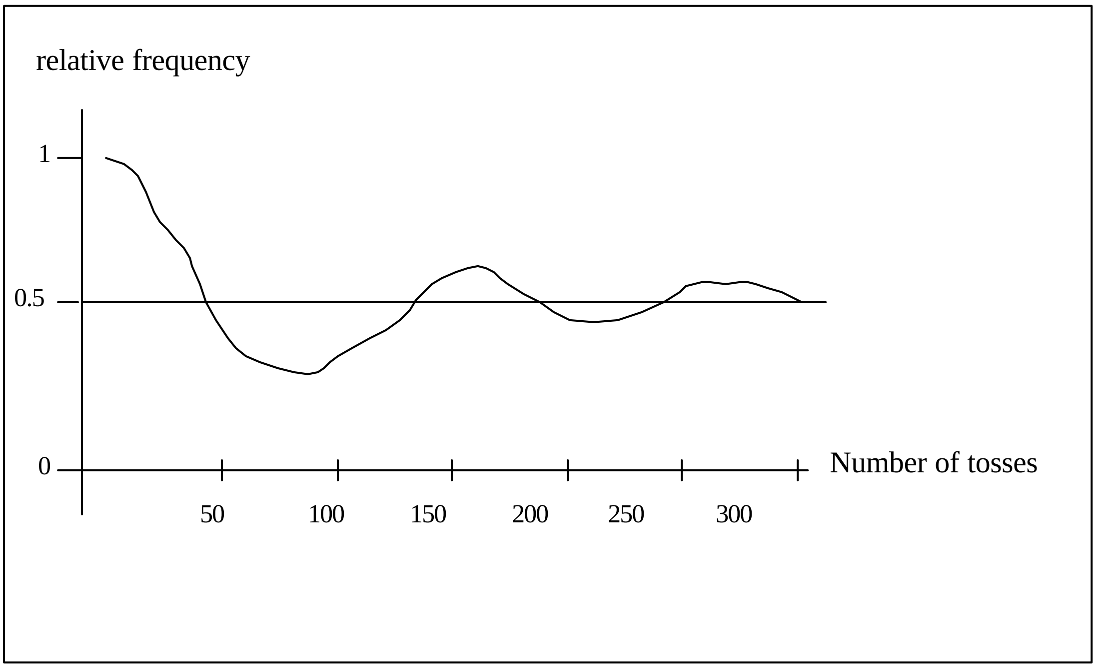
Therefore, probability of an even number = $\frac{3}{6} = \frac{1}{2} = 0.5$

2. Relative Frequency Approach (Empirical Concept)

Probability of an event = the ***proportion of times*** an event occurs in the long run under uniform condition

The difficult with the relatively frequency approach is that people often use it without evaluating a *sufficient number of outcomes*.

e.g. 9 By tossing a fair coin 300 times. We can see that although the proportion of head was far from 0.5 in the first 100 tosses, it seemed to stabilize and approach 0.5 as the number of tossed increased.



3. Subjective Approach

Probability of an event = the degree of belief or degree of confidence placed in the occurrence of the event *by a particular individual* based on the evidence available.

e.g.10 A judge is deciding whether to allow the construction of a nuclear power plant on a site where there is some evidence of a geological fault. The fact that there is no relative frequency of occurrence evidence of pervious accidents at this location does not excuse him from making a decision.

He must ask himself the question, " What is the probability of a major nuclear accident at this location?" He must use his best judgment in trying to determine the subjective probabilities of a nuclear accident.

Principles of Counting

If an experiment contains a large number of outcomes, it may be difficult to count the number of outcomes in an event.

Counting Rule 1: If any one of k mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is:

$$\mathbf{k^n}$$

e.g.11 If a die (having six sides) is rolled twice, the number of different outcomes is:
{1,1 1,2 1,3 ....2,1 2,2 ...... 2,6 3,1 3,2 ....6,3 6,4 6,5 6,6}

Sol] $n = 2, k = 6$
the number of possible outcome = $k^n = 6^2 = 36$

e.g.12 If there are ten multiple choice questions on a exam, each having three possible answers, how many different possibilities are there in terms of the sequence of correct answers?

Sol] n = , k =
the number of possible outcome = k^n = =

Counting Rule 2: If there are k_1 mutually exclusive and collectively exhaustive events on the first trial, k_2 events on the second trial, ..., and a k_n events on the n^{th} trial, then the number of possible outcomes is:

$$\mathbf{(k_1)\ (k_2)\ \ldots\ (k_n)}$$

e.g.13 A particular brand of women' s jean can be ordered in seven different sizes, three different colors, and three different styles. How many different jeans would have to be ordered if a store wanted to have one pair of each type?

Sol] $k_1 = 7, k_2 = 3, k_3 = 3$

the number of possible outcome = $k_1 \times k_2 \times k_3$
$= 7 \times 3 \times 3 = 63$

e.g.14 A car license plate consisted of two letters followed by four digits, what is the number of all possible outcomes?

Sol] k_1 = , k_2 = , k_3 = , , , .

the number of possible outcome = $k_1 \times k_2 \times k_3 \times$
= =

Counting Rule 3: The number of ways that all n objects can be arranged in ordered be:

$$n! = n(n-1)(n-2)...(2)(1)$$

(remark : 0! is defined as 1)

e.g.15 If each letter is used once, how many different four-letter "words" can be made from the letters E, L, O, and V.

Sol] n = 4,
the number of possible outcome = n! = 4!
= 4 x 3 x 2 x 1 = 24

e.g.16 If a set of six textbooks is to be placed on a shelf, how can we determine the number of ways in which the six books may be arranged?

Sol] n = ,
the number of possible outcome = n! = !
= =

Counting Rule 4: Permutations

The number of ways of arranging r objects selected from n objects <u>in order</u> is:

$$_nP_r = \frac{n!}{(n - r)!}$$

Note : Order will make a difference.

e.g.17 The Big Triple at the local racetrack consists of picking the correct order of finish of the first three horses on the ninth race. If there are 12 horses entered in today's ninth race, how many Big Triple outcomes are there?

Sol] n = 12, r = 3
the number of possible outcome = $_nP_r$ = n! / (n-r)!
= $_{12}P_3$ = 12! / (12-3)!
= 12! / 9!
= 1320

e.g.18 A basketball team must schedule a game with each of three different teams (Team A, B and C). There are five different dates available for games. How many ways different schedules can be made?

Sol] n = , r =
the number of possible outcome = $_nP_r$ = n! / (n-r)!
= P =
=
=
=

Counting Rule 5: Combinations

The number of ways of selecting r objects out of n objects, ***irrespective of order*** is:

$$_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

e.g.19 A student has seven books that he would like to place on a shelf. However, only four books can fit into the shelf, how many ways are there of placing four books on the shelf?

Sol] n = 7, r = 4
the number of possible outcome = $_{n}C_{r}$ = n! / r!(n-r)!
= $_{7}C_{4}$ = 7! / 4! (7 - 4)!
= 7! / 4! 3!
= 35

e.g.20 A reading list of articles for a course contains 20 articles. How many ways are there to choose three articles from the list?

Sol] n = , r =
the number of possible outcome = $_{n}C_{r}$ = n! / r!(n-r)!
= C =
=
=

Class Exercise 1:

A teacher has four projects - A, B, C, D - to assign to his final year students. Each project will be assigned to one student.

If the teacher has four final year students : Annie, Betty, Cathy and David,

a. how many different arrangements are possible?
b. what is the probability that project D is assigned to Betty?

Probability Rules

1. **0 £ P(A) £ 1** for any event A

2. **P(S) = 1** S is the sample space

3. **P(A) + P(A') = 1**
   or, P(A') = 1 - P(A) for any event A

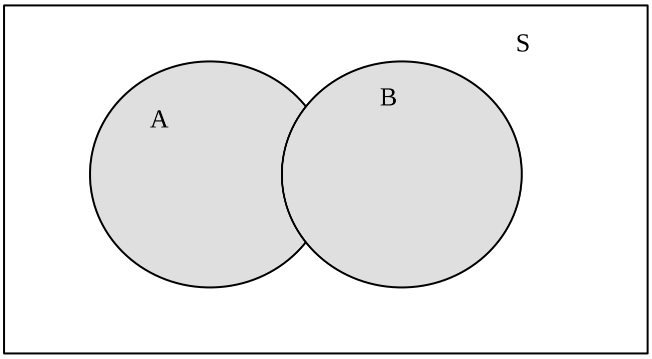
e.g.21 The probability that Bob will finish his paper is 3/7. Find the probability that he will not finish his paper?

Sol] Let A represents that Bob will finish his paper
then $\overline{A}$ represents that Bob will not finish his paper.

P(A) = 3/7, $P(\overline{A})$ = 1 - P(A)
= 1 - 3/7 = 4/7

4. Addition Rule:

P(A or B)
= P(AÈB) = P(A) + P(B) - P(AÇB) *for A & B are not mutually exclusive*



A and B are not mutually exclusive, $P(A \cap B) \neq 0$

e.g.22 If a single card is drawn from an ordinary deck of playing cards, find the probability that it will be red or a face card (jack, queen, or king).

Sol] Let A represents drawing a red card,
Let B represents drawing a face card.

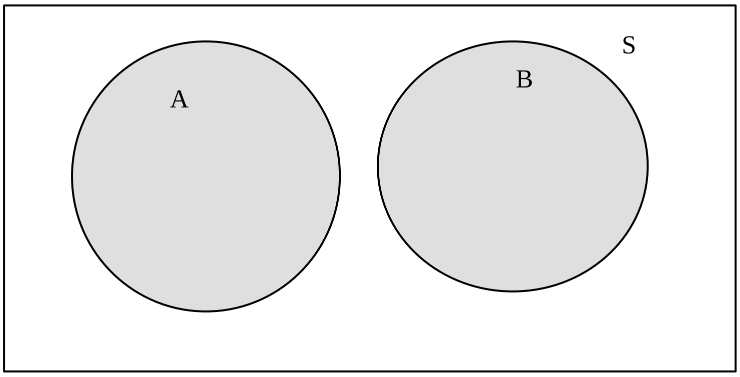
$P(A) = 26/52$; $P(B) = 12/52$; $P(A \cap B) = 6/52$;

Using the addition rule,

Prob.(red or face card) =
$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 26/52 + 12/52 - 6/52$
$= 32/52$
$= 8/13$

Note that: if A & B are mutually exclusive, then $P(A \cap B) = 0$

$P(A \text{ or } B)$
$= P(A \cup B) = P(A) + P(B)$ *for A & B are mutually exclusive*



A and B are mutually exclusive, $P(A \cap B) = 0$

In general, if $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events, then

$$\mathbf{P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)}$$
[proof] : ??

e.g.23 If the probabilities are respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue. what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

Sol] Let G, W, R, and B be the events that a buyer selects a green, white, red, or blue automobile respectively.

$P(G) = 0.09;$ $P(W) = 0.15;$ $P(R) = 0.21;$ $P(B) = 0.23;$

since the four events are mutually exclusive, the probability is

$$
\begin{aligned}
P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\
&= 0.09 + 0.15 + 0.21 + 0.23 \\
&= 0.68
\end{aligned}
$$

5. Conditional Probabilities:

The probability of event A occurring when we know that event B *has already occurred* is called Conditional Probability, and is written as **P(A|B).**

P(A|B) means the probability that event A will occur , given the condition that the event B has occurred, or simply **the probability of A given B**.

e.g.24 Two balls are drawn without replacement from a bag containing 3 white and 2 black balls. Find the probability that

a. the second ball is white given that the first ball is black?
b. the second ball is white given that the first ball is white?

Sol] Let W represents white ball is drawn,
Let B represents black ball is drawn.

a. $P(2^{nd} \text{ is } W \mid \mathbf{1^{st} \text{ is } B}) = P(W|B) = 3/4$

b. $P(2^{nd} \text{ is } W \mid \mathbf{1^{st} \text{ is } W}) = P(W|W) = 2/4$

e.g.25 A study was undertaken at a certain college to determine what relationship, if any, exists between Maths. ability and interest in Maths. The ability and interest for 150 students were determined, with the results in the following table:

Ability	Interest			Total
	Low (LI)	Average (AI)	High (HI)	
Low (LA)	40	8	12	60
Average(AA)	15	17	18	50
High (HA)	5	10	25	40
Total	60	35	55	150

If one of the particulars in the study is chosen at random, what is the probability

a. of selecting a person who has low interest in Maths.?
b. of selecting a person with average ability?
c. that the person has high ability in Maths. given that the person selected has high interest in Maths.?
d. that the person has high interest in Maths. given that the person selected has average ability in Maths.?

Sol]

a. since there are 60 participants with low interest out of a total of 150, the probability is 60/150 = 2/5.

b. since there are 50 participants with average ability out of a total of 150, the probability is 50/150 = 1/3.

c. of the 55 participants with high interest, 25 have high ability. Therefore, the probability is 25/55 = 5/11.

d. since 50 participants have average ability and of these, 18 have high interest, the probability is 18/50 = 9/25.

or use Conditional Probability Formula:

P(A|B) = P(AÇB)/P(B)

as in part (c),
p(high ability | high interest)
= p(high ability ∩ high interest) / P(high interest)
= (25/150) / (55/150) = 5/11

as in part (d),
p(high interest | average ability)
= p(high interest ∩ average ability) / p(average ability)
= (18/150) / (50/150) = 9/25

6. Independent Events:

If E and F are events such that the occurrence of F in no way influences the occurrence of E, then E and F are called Independent Events.

E and F are independent events if the probability of E occurring given that event F has occurred is identically equal to the probability of event E occurring.

E and F are independent events if

P(E|F) = P(E) , or

P(F|E) = P(F)

e.g.26 Based on the information provided in example 25, determine whether ability in Maths. is statistically independent of interest in Maths.

P(HI) = 55/150; P(HI|HA) = P(HI∩HA)/P(HA)
= (25/150) / (40/150)
= 25/40 = 5/8

If ability and interest are independent, then P(HI|HA) = P(HI), but

P(HI|HA) = 5/8
¹ 55/150 = **P(HI**)

Therefore, ability in Maths. and interest in Maths. should not be statistically independent.

7. Multiplication Rule:

by the conditional probability formula, (**Joint Probability**)

P (A and B)

= P(AÇB) = P(A|B) P(B) = P(B) P(A|B) *if A and B are not independent*
= P(BÇA) = P(B|A) P(A) = P(A) P(B|A) *if A and B are not independent*

e.g.27 If two balls are drawn without replacement from a bag containing 3 red, 2 black, and 1 white ball, what is the probability of getting 2 red balls?

Sol] Let R represents drawing red ball

P(getting 2 red balls)
= P(R∩R) = P(R) P(R|R)
= 3/6 x 2/5 = 1/5

However, **if A and B are independent**, P(A|B) = P(A), and P(B|A) = P(B), then

P(A and B)
= **P(AÇB)** = P(A|B) P(B) **= P(A) P(B)**
= **P(BÇA)** = P(B|A) P(A) **= P(B) P(A) = P(A) P(B)**

e.g.28 The probability that Tom will pass in Statistics is 0.5, and Mary will pass in Statistics is 0.8. What is the probability that both Tom and Mary will pass in Statistics?

Sol] Let T represents Tom pass in statistics,
Let M represents Mary pass in statistics

P(T) = 0.5 ; P(M) = 0.8

P(Tom and Mary pass)
= P(T∩M)
= P(T) x P(M) assume *events T and M are* ***independent***
= 0.5 x 0.8 = 0.4

In general, if A_1, A_2, A_3,..., A_n are statistically independent, then

P(A_1ÇA_2ÇA_3Ç...ÇA_n) = P(A_1) x P(A_2) x P(A_3) x ... x P(A_n)

[proof] : ???

Note : as a general rule, sampling with replacement assures that two events will be independent, whereas sampling without replacement produces two events that will be dependent events.

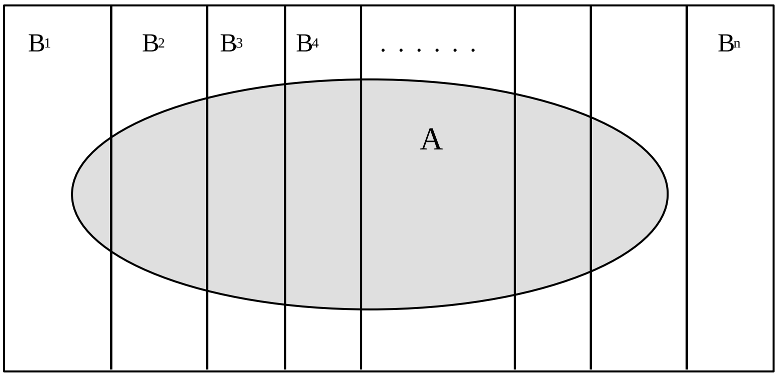
8. Law of Total Probabilities:

Suppose that the sample space S consists *n mutually exclusive* and *collectively exhaustive* events, B_1, B_2, ..., B_n , then the probability of any event A, consists of the joint probability of event A occurring with event B_1, and the joint probability of event A occurring with event B_2, and up to the joint probability of event A occurring with event B_n.

P(A) = P(AÇB$_1$) + P(AÇB$_2$) + ... + P(AÇB$_n$)

= P(A|B$_1$)P(B$_1$) + P(A|B$_2$)P(B$_2$) +...+ P(A|B$_n$)P(B$_n$)

Sample Space S



e.g.29 A manufacturing plant is operated in 3 shifts, morning shift, day shift and night shift. If 30% of the total production is produced during morning shift, 50% is produced during day shift, and 20% is produced during night shift. The defective percentage during the 3 shift are 5%, 2% and 8% respectively. What is the total percentage of defective over the whole production?

Sol] Let A represents the defective product;
B_1 represents the morning shift production;
B_2 represents the day shift production; and
B_3 represents the night shift production.

$P(A\|B_1) = 5\%$;	$P(A\|B_2) = 2\%$;	$P(A\|B_3) = 8\%$;
$P(B_1) = 30\%$;	$P(B_2) = 50\%$;	$P(B_3) = 20\%$.

$P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) + P(A|B_3)\,P(B_3)$
$= 0.05 \times 0.3 + 0.02 \times 0.5 + 0.08 \times 0.2$
$= 0.0410 = 4.10\%$

9. Bayes' Theorem:

Bayes' Theorem offers a powerful statistical method of evaluating new information and revising our prior estimates of the probability that things are in one state or another.

The basic formula for conditional probability under dependence is called Bayes' Theorem.

$$\mathbf{P(A|B) = \frac{P(A\ Ç\ B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}}$$

e.g.30 Assume that we have equal numbers of two types of deformed dice in a bowl. For Type 1 deformed dice, even number comes up 40% of the time, and for Type 2 deformed dice, even number comes up 70% of the time.

If a die is randomly selected from the bowl, what is the probability that it is a Type 1 die?
Prob(Type 1) = 0.5

Now, addition information is available. One die is drawn, rolled once, and comes up even numbers. What is the probability that it is a Type 1 die?

Sol]

Elementary Event	P(Elementary Event)	P(Even|Elementary Event)	P(Even $\cap$ Elementary Event)
Type 1	P(Type 1) = 0.5	P(Even|Type 1) = 0.4	P(Even $\cap$ Type 1) = 0.4 x 0.5 = 0.20
Type 2	P(Type 2) = 0.5	P(Even|Type 2) = 0.7	P(Even $\cap$ Type 2) = 0.7 x 0.5 = 0.35
	1.0		P(Even)= 0.55

P(Type 1 | Even) = P(Type 1 $\cap$ Even) / P(Even)
= 0.2 / 0.55
= 0.3636

We may feel that one roll of the die is not sufficient to indicate its characteristics. In this case, we can obtain additional information by rolling the die again.

Assume that the same die is rolled a second time and again comes up even. What is the further revised probability that the die is Type 1?

Sol]

Elementary Event	P(Elementary Event)	P(Even | Elementary Event)	P(2 even | Elementary Event)	P(2 Even ∩ Elementary Event)
Type 1	0.5	0.4	0.4 x 0.4 = 0.16	0.16 x 0.5 = 0.080
Type 2	0.5	0.7	0.7 x 0.7 = 0.49	0.49 x 0.5 = 0.245
	1.0			P(2 Even)= 0.3250

P(Type 1 | 2 even) = P(Type 1 ∩ 2 even) / P(2 even)
= /
=

e.g.31 Assume that the same die is rolled three times, and the first two rolls come up even and the third roll comes up odd. What is the further revised probability that the die is Type 1?

Sol]

Elementary Event	P(Event)	P(Even | Event)	P(Odd | Event)	P(E,E,O | Event)	P(E,E,O ∩ Event)
Type 1					
Type 2					

P(Type 1 |) = P(Type 1 ∩) / P( )
=
=

e.g.32 For example 29, the quality controller found a defective product during the quality control process, find the probability that this defective product was produced during morning shift.

Sol] The probability for a given defective product which is produced during morning shift is $P(B_1|A)$,

$P(A|B_1) = 5\%$; $P(A|B_2) = 2\%$; $P(A|B_3) = 8\%$;
$P(B_1) = 30\%$; $P(B_2) = 50\%$; $P(B_3) = 20\%$;
$P(A) = 4.1\%$

$$\mathbf{P(B_1|A)} = P(B_1 \cap A)/P(A)$$
$$= P(A|B_1)P(B_1)/P(A)$$
$$= 0.05 \times 0.3 / 0.041$$
$$\mathbf{= 0.3659}$$

e.g.33 What are $P(B_2|A)$ and $P(B_3|A)$?

Sol] $\mathbf{P(B_2|A)} =$
=
=
=

$\mathbf{P(B_3|A)} =$
=
=
=

Class Exercise 2:

A manager has three clerks working for him - Lucy, May and Nancy. May and Nancy are responsible for 30% and 50% of the typing work respectively. For those documents typed by Lucy, 8% of them contains typing errors. For those documents typed by May, 10% of them contains typing errors. For those documents typed by Nancy, 5% of them contains typing errors.

a. What is the probability that a randomly selected document which contains typing error?
b. What is the probability that a document with typing errors was typed by May?

Class : ____________ **Name** : __________________ **No.** : __________

Class Exercise 1 (Solution)

a) Number of possible arrangements of assignments
= !
=

b) Number of arrangements which means project D is assigned to Betty
= !
=

P(project D is assigned to Betty) = / =

Class : ____________ **Name** : __________________ **No.** : ___________

Class Exercise 2 (Solution)

Let L be the event that the document is typed by Lucy
N be the event that the document is typed by Nancy
M be the event that the document is typed by May
E be the event that the document contains error.

P(N) = P(M) =

P(E|L) = P(E|N) = P(E|M) =

P(L) = 1 - P( ) - P( ) = 1 - - =

a) P(E) = P(E and L) + P(E and N) + P(E and M)
= P(E|L) P(L) + P(E|N) P(N) + P(E|M) P(M)
= x + x + x
= *(4 decimal places)*

b) P(M|E)
= P(and) / P( )
= P(|) P( ) / P( )
= x /
= *(4 decimal places)*