Multiple representations of addition and subtraction-related problems by third, fourth and fifth graders

PROJECT PROPOSER

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CHAPTER ONE

Introduction

This chapter deals with five major subheadings: context of the project, purpose of the project, research hypotheses, interpretative framework and research design. Firstly, the theoretical and practical significance of the research problem - a preliminary means of locating the problem of investigation - will be considered under the subheading of context of the project. Secondly, the purpose of the project will be discussed focusing the main problem of the research as to assess the student-generated multiple representations of addition and subtraction-related problems. Thirdly, since research hypotheses are the bases for developing a set of procedure for conducting a research project, three-research hypotheses will be discussed linking with their theoretical underpinning. Fourthly, a summary of main theoretical aspects - constructivism, constructionism, Vygotskian and Piagetian perspective and ethno-mathematics - will be examined under the subheading of interpretative framework. Fifthly, the rational of research method, triangulation, research questions, sources of data, data collection techniques, procedure of obtaining data, pilot study and resources will be discussed under the subsection of research design.

Context of the proposed research

Generally, the term *representation* refers to the process of construction, abstraction and demonstration of mathematical knowledge. Representation of mathematical concepts, principles, and problem situation is one of the issues of mathematics learning. Particularly, it has become an area of research since 80s. As a result, the National Council of Teachers of Mathematics in the United States has decided that representation will be a new "process standard" (Pape & Tchoshanov, 2001, p. 122) rather than simply a part of communication as it was in 1989. Moreover, the context of the proposed research is sought to discuss from two perspectives: practical and theoretical significance, which are elucidated below.

Practical significance. The present research problem is a consequence of my professional activity as a primary, secondary and tertiary level teacher as well as a Teacher Educator. I have been observing for seven years that student-generated representation of mathematical concepts, problems and symbols are not objective: The subjectivity of representation can lead some theoretical underpinning. As this observation relates to the perspective of constructivism in mathematics learning, the present research would be a useful means for resolving the problems of mathematics teaching and learning. Specifically, the present research would be significant in four ways: Firstly, since the present research is a case study of nine students studying at primary level, it can develop a new perspective on learner-generated representational system (Kamii, Kirkland, & Lewis, 2001) of mathematical problems. Secondly, the representational system of addition and subtraction-related problem would be helpful in deriving some implications for classroom learning. Thirdly, the present research would be useful for contributing to identify such representational system that can help teachers to design the teaching/learning activities effectively. Fourthly, as this area is a recent theoretical construct of constructivism (Kaput, 1999), the role of representation in mathematics learning is still to develop and construct. This research can contribute to the development of a theoretical construct in this regard.

Theoretical significance. Ideas about representation in research and teaching and learning of mathematics have evolved considerably in recent years (Goldin & Shteingold, 2001; Moritz, 2000). The National Council of Teachers of Mathematics in the United States has selected representation as a process standard of school mathematics in 2000 (Fennell & Rowan, 2001), due to its increasing importance in teaching and learning mathematics whereas it was discussed as a part of a communication standard in 1989 (Fennell & Rowan, 2001).

In order to discuss the context of the proposed research problem, it is very important to explicate the term "representation". Representation can be viewed as internal-abstraction of mathematical ideas or cognitive schemata (Pape & Tchoshanov, 2001). Such schemata are constructed by the learners in order to establish them as a part of their internal mental network (Hiebert & Carpenter, 1992). Furthermore, the role of representation in probing understanding of mathematics learning is vital because understanding of learning is possible only when the concept, knowledge, formula or principle becomes a part of a person's network of representation (Hiebert & Carpenter, 1992).

Bruner (1966) distinguished three different modes of mental representation – the *sensory-motor (enactic)*, the *iconic* and the *symbolic* (Tall, 1994). The sensory motor representation is performed through action; the iconic representation is carried out through images; and the symbolic representation is carried out through mathematical languages and symbols (Bruner, cited in Tall, 1994). This shows a hierarchy of a representational system whereas Vygotsky (1935) emphasizes on children-built representation rather than conventional symbols in the beginning (Sierpinska, 1998) which would be developed into a perfect symbolization through cognitive apprenticeship. According to Piaget (1977) the child represents his/her ideas about the reality, but not reality itself (Kamii et al., 2001).

Representation has emerged as a part of mathematical communication (Pape & Tchoshanov, 2001). According to Pirie (1998) the means of mathematical communications can be classified as ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions and quasi-mathematical language. The function of any types of representation is to communicate the mathematical ideas (Pirie, 1998). Consequently, the appropriate representational system helps for effective communication of mathematical ideas.

It is very interesting to note that a single mathematical concept can be represented to convey different meanings (Pirie, 1998). For instance, three quarters represents a number while three items out of four is a mental image (or a way of drawing). Similarly three over four (3/4) is a way of writing and symbolizing whereas three divided by four is a process. Multiple sources of communication is a means of developing the understanding of mathematics learning (Pirie, 1998).

There are three types of representation: internal, shared and external. External representation is, for example, a sign or configuration of signs, characters or objects

(Goldin & Shteingold, 2001). Most of the external representations are produced by the students rather they construct them because they are objectively defined and decided. The external and internal representations constructed through shared mode, are called shared representations. Such types of representations can be found in classroom discourse and communal situation (Gergen, 1995). For instance, the symbol of addition (+) can be constructed through an interaction between the learner and teacher. The internal representations deal with students' changing attitudes towards mathematics or mathematical concepts (Goldin & Shteingold, 2001). Furthermore, they are also called psychological representations in which different representational systems can be found as verbal/syntactic, imagistic, formal notational and affective (Goldin & Shteingold, 2001).

The focus of constructivism is in internal and shared representations because the learner constructs them. Internal representation also consists of affective representation which can help develop the understanding of learning (Goldin & Shteingold, 2001). On the contrary, the behaviourists have not included the concept of internal and shared representations rather they have focused on demonstration of external representation in order to measure the students' performance. The classroom goals such as "behavioural objectives " are the best examples of the preferences of behaviourists (Sund & Picard, cited in Goldin & Shteingold, 2001).

The emphasis of research in the area of representation has been found on children generated representation and their subsequent impact in learning mathematical concepts (Lowrie, 2001; Swafford & Langrall, 2000; Verschaffel, 1994). A number of research reports have been reviewed in order to build up the present research problem. The details of reviewed research for example, problem representation (Lowrie, 2001; Swafford & Langrall, 2000), graphical representation (Moritz, 2000), diagrammatic representation (Diezmann, 1999), numerical (symbolic) -situation representation (Outhred & Saradelich, 1997), representation by retelling the addition and subtraction problem (Verschaffel, 1994) representation of solution of addition problems by using analogs (Boulton-Lewis & Tait, 1993) are discussed in chapter 2.

Purpose of the project

The proposed research will focus on assessing student-generated representation of addition and subtraction-related problems at primary level.

Specifically, the central problem of this research is to identify the effectiveness of student-generated representation of addition and subtraction-related problems. For this, the research is sought to investigate three major aspects of representation of addition and subtraction-related problems: First, it deals with the level of student-generated representation of addition and subtraction problem. Furthermore, it aims at comparing each student's three different representation (verbal, diagrammatic, numerical (symbolic)) of each problem. Secondly, the representation of addition-related problems is compared to that of the subtraction-related problems in order to identify the difference in representing those problems. Thirdly, the research seeks to identify the effective means of representation of addition and subtraction-related problems.

Research hypotheses

Three- research hypotheses have been formulated in order to answer the research questions. These are discussed as follows:

- The level of representation of each of the participants would be the same in all three representational systems.
- The addition problem would be represented as effective as the subtraction problem.
- Diagrammatic representation would be a more effective means of representation than verbal and numerical (symbolic) means of representation of addition and subtraction-related problems.

As Hiebert (1992) said that understanding of mathematics learning is the representation of knowledge by internal mental network of learner, the knowledge would be represented in the same manner whatever be the representational system. In this light, the first hypothesis has been formulated to assess whether the three-representations (verbal, diagrammatic and numerical (symbolic)) of a problem made by individual students would be the same. Furthermore, it is also based on the perspective of Luria and Vygotsky (1977) as representation actually equates the thought (Fosnot, 1996). Theoretically, the level of individual students' representation may be the same regardless of the representational system. Similarly, the second hypothesis is based on the findings of the research on "retelling" the addition and subtraction problem (Verschaffel, 1994, p. 141), as there is no difference in representing addition and subtraction problem provided that the language (either consistent or inconsistent) of the problem should be the same. Similarly, according to Fuson (1992) the representation of addition and subtraction problems is theoretically same. In this light, the second hypotheses would try to discuss the difference (if any) in representing addition and subtraction problems. The third hypothesis is theoretically closer to the framework and findings of Diezmann (1999) and Moritz (2000) in which diagrammatic representation has been discussed as an important means of learning mathematics.

Interpretative framework

The research is proposed to be a case study of nine primary schoolchildren regarding their representation of addition and subtraction-related problems by means of diagrammatic, and verbal numerical (symbolic) representation in which the theoretical background of representation is derived from constructivism, constructionism and ethnomathematics. On the one hand, constructivism deals with mental construction and representation of knowledge by virtue of experiential world (Cobb, 1996; Confrey, 1995; vonGlasersfeld, 1995). This model suggests that learning is a process of construction and representation of knowledge (Cobb, 1996; Confrey, 1995; vonGlasersfeld, 1995). The focus of representation according to constructivism is internal and subjective (Cobb, 1996; Confrey, 1995; vonGlasersfeld, 1995). On the other hand, constructionism deals with the shared representation, which exists in the shared (interactive-discourse) mode (Gergen, 1995). It further emphasizes that the construction of meaning and its representation is societal rather than individual.

Representation is a general context of ethno-mathematics (Ubiratan D' Ambrosio, personal communication via email, May 26, 2002). Generally, mathematical knowledge has been represented through cultural myths, artefacts, language, and other means of communication.

In the light of above-mentioned theoretical construct, representation is an essential aspect of mathematics learning. It is a means of constructing, organising and presenting mathematical ideas. Details of different models will be discussed in chapter 2.

Research design

Rationale of research method. The proposed research will adopt a qualitative research method. Particularly, it is a case study of student-generated multiple representations of addition and subtraction-related problems. Generally, a case study is carried out in order to explore the interpretative and subjective dimensions of educational phenomena (Cohen & Manion, 1992). Since the representation of mathematical content, concept, and context is subjective (Goldin & Shteingold, 2001) and a multifaceted idea, the investigation of such area can be done effectively through case study methods (Burns, 2000). Furthermore, such cases can contribute to the extension of a theory by supporting the existing principle or by challenging it through the outcome of case studies (Burns, 2000; Merrian, 1988)

Triangulation. Triangulation is a method of establishing internal validity of a qualitative research. It may be defined as the use of two or more methods of data collection in the study of some aspect of human behaviour (Burns, 2000; Cohen & Manion, 1992). Broadly speaking, there are four types of triangulation: data triangulation including time, space, and person, investigator triangulation, theory triangulation and methodological triangulation.

The proposed study will consider the data, methodological, and theory triangulation. Data triangulation will be considered by collecting the data from different person (students) in different time (two times). Similarly, the methodological triangulation is apparent to this study, since three different methods of data collection will be used in order to identify the students' representation. The theory triangulation is related to an epistemological and ontological justification (Merrian, 1988) of the term representation. Furthermore, a perspective on representation will be discussed from different schools of learning theories. For instance, constructivism focuses internal and affective representational system whereas the constructionism focuses on shared one. Furthermore, the Vygotskian perspective is to develop a representation through interaction and as a function of language whereas the Piagetian perspective is to construct the internalised schemata (internal representation) of knowledge.

Research questions. The proposed research has intended to answer the following three questions:

- Is the level of representation of each participant likely to be the same in all three representational systems?
- To what extent is the student-generated representation of addition problem different from that of the subtraction problem?
- Which representational system is the effective means of representation for addition and subtraction problem?

Sources of data. The main sources of data will be the third-, fourth- and fifth-grade students studying in one of the Western Australian schools. Three students from each grade level will be selected purposively (Cohen & Manion, 1992) including the high and medium achiever in mathematics. The level of achievement will be distinguished as per the teacher's assessment.

Data Collection Techniques. Firstly, three different sets of problem-solving task (PST) will be developed after discussing with the schoolteachers who are teaching in the third-, fourth- and fifth-grade level. Basically, this instrument will be prepared in order to identify the student-generated diagrammatic and numerical (symbolic) representation. Secondly, an interview questionnaire will be devised in order to investigate the student-generated verbal representation. Generally, the tools of data collection will be developed in three steps: First, the objectives of the tools will be formulated; secondly, a try-out is conducted whether the problem-solving task and questionnaires are appropriate to seek the intended information; and thirdly, necessary changes will be made as per the result of try-out.

Case Study Protocol. Case study protocol is a set of rules and procedures that should be followed in the study (Burns, 2000). Regarding the present study a protocol will be prepared for preceding the present study. Such protocol will include the purpose of the study, the issues, the setting, the propositions being investigated the letter of introduction, review of theoretical basis, operational procedures for getting data, sources of information, questions and lines of questioning, guidelines for report, relevant readings and references.

Procedures of obtaining Data. One of the government primary schools in Bentley suburb will be identified within July 2002. Consent from the school, parents, and teachers is essential in order to conduct the study. For this, a written contract will be made with the respective stakeholders.

Right of anonymity of each student will be respected. For this, pseudonym of each student will be used in the process of analysing the data. The data will be kept confidential and will not be transferred to any other agencies.

The investigator will visit the respective schools and interview the students. The interview will be about the problem-solving task (PST). It will be conducted in two different times: before and after solving the problems for numerical (symbolic) and diagrammatic representation. The interview will be audio tapped.

Pilot study. The pilot study will be conducted by the end of July 2002 at a government primary school situated in Bentley. The purpose of this study is to validate the problem-solving task (PST) and interview questionnaire.

Resources. The Science and Mathematics Education Centre (SMEC) will provide the researcher with a shared study desk and a computer, which is networked for Internet connection. The researcher has access to the required stationery and photocopying facilities at SMEC. An audiotape will be required in order to record the interview with the participants.

Regarding the human resources, the researcher will be sufficient to conduct this research. Furthermore, he will discuss with the supervisor if there are such issues to resolve. As concern to time, a six-month from July and onwards is essential for this study.

CHAPTER TWO

Literature Review

Introduction

This chapter deals with a review of related literature in the area of representation. Particularly, two types of literature will be reviewed: Firstly, three different perspectives as constructivism, constructionism and ethno-mathematics will be discussed to draw the inference about the role and nature of representation in mathematics learning. Secondly, a descriptive survey of the research related to representation of mathematical contents concepts and problems will be carried out in order to develop a model for the proposed research problem.

The meaning of representation

There are four main ideas used to conceptualise the notion of representation. Firstly, within the domain of mathematics, representation can be considered as an internal abstraction of mathematical ideas or cognitive schemata that are developed by the learner through experience (Pape & Tchoshanov, 2001). Secondly, representation can be explicated as mental reproduction of a former mental sate (Seeger, cited in Pape & Tchoshanov, 2001). Thirdly it refers to a structurally equivalent presentation through pictures, symbols and signs (Seeger, cited in Pape & Tchoshanov, 2001). Lastly, it is also known as something in place of something (Seeger, cited in Pape & Tchoshanov, 2001)

There is no unanimity in representing the term representation. It can also be used to mean hypothesised mental constructs and material notations (Goldin & Shteingold, 2001). The former is an internal representation while the second is external ones (Kaput, 1999). Cifarrelli uses the word "representation" exclusively as mental representation whereas Evan used this word as material representation (cited in Kaput, 1999). Similarly "fusion" is referred to emphasise on maintaining structure and orientation in time and in the space of actions and possibilities surrounded by a symbol rich with experience (Kaput, 1999)-internalising the external representation. Representational capacity of early men has been believed to have begun about 1.5 million years ago in the form of mimetic (Donald, 1991) .

According to the perspective of Nunes, there are two types representation: The first is compressed and the second is extended. If a child represents all numbers by analogs for the operation of (2 + 3), then the representation is extended. On the contrary, if it is represented either only two or three (for example, counting starts from four and ends at five or it starts from three and ends at five), it is known as compressed (Nunes, 1997).

It is interesting to note that the Indo-Arabic number system is an example of compressed representation, while many of the others in the past had been extended representations. The Egyptian numeration system (Eves, 1969a) is one of the examples of such type of numeration system

There are various examples in mathematics that help visualize the compressed and extended representations. The "counting situation as (1,2,3,4,5) →5" (Nunes, 1997, p.37) is very simple but meaningful compressed representation. Furthermore, the example as 123→(100+20+3) is an extended representation.

According to Vergnaud (1997) representation is an attribute of mathematical concepts as they defined the mathematical concepts by a set {S, I, R} where, S=the set of situation that make concept useful and meaningful; I= the set of operational invariants that can be used by individuals to deal with these situations; R=the set of symbolic, linguistic, graphic or gestural representation that can be used to represent invariants situations and procedures (Vergnaud, 1997).

Looking at it from the etymological point of view, representation is to identify, select and present something for something. For instance, in order to represent "five", the learner can select five tally bars or five unit cubes or five any of the objects.

The term representation can be viewed as "presentation" and "re-presentation". However, if representation is regarded as presentation and re-presentation then mathematics learning-process will be oriented to reproduction of the ideas (Confrey, 1995). In fact, representation is a part of the process of construction of knowledge, which can be performed either by sharing the ideas between two or more people or by constructing individually (Kamii et al., 2001). In this light, we may consider the lower level of representation is to present or re-present the mathematical ideas per se.

Constructivism, Constructionism and Representation

Constructivism has been widely accepted in mathematics and science learning since 80s (Thompson, 1995). Education suffered a decline in the last 20 or 30 years (vonGlaserfled, 1995), which led to search for a different view of the learning process. In fact, the central idea of constructivism is to learn by constructing the knowledge rather than receiving from the teacher. The perennial concept regards the knowledge as an independent entity of the world, which does not help, for developing understanding of learning (vonGlasersfeld, 1995). Furthermore, the constructivist approach of knowing is to construct for an active representation of reality and to develop it as a part of an internal mental network (Hiebert & Carpenter, 1992) of the learner.

Looking at the historical perspective, we can find that the scepticism emerged in 500 BC (Eves, 1969a) which did not accept the process of logical representation of truth (vonGlasersfeld, 1995). However, sceptics did not suggest an alternative way of representing knowledge instead of reiterating the argument to oppose the rationalism- the doctrine that knowledge is acquired and represented by reason without resort to experience (vonGlasersfeld, 1995). The knowledge which represents the real world, particularly, the experiential world through which one can relate with the abstract knowledge (vonGlasersfeld, 1995). The behaviourist's concept of adaptation (vonGlasersfeld, 1995) does not see any difference between human beings and other animals. However, for the constructivism, the most important aspect is that the customary conception of truth as the correct representation of states or events of an external worlds was replaced by the notion of the viability (vonGlasersfeld, 1995). Here, viability implies the adequacy in the contexts in which the concept, models and theories are created (vonGlasersfeld, 1995) and represented. Moreover, constructivism in itself is a process of meaning making through representation that results in reflective abstractions, producing symbols within a medium (Cobb, 1996).

Generally, there are two views of knowledge known as exogenic and endogenic in which the previous deals with world-centred and the later deals with mind-centred knowledge (Gergen, 1995). In general, the views on representation of knowledge fall in the continuum of psychological-material reality. The exogenic tradition regards the external materials are the given while the endogenic regards the internal mental state is given (Gergen, 1995). The two different systems of representation of knowledge has a fundamental difference in viewing the learner in which the exogenic prefers to specify the learner as a "tabula rasa" while the endogenic focuses on rational capacities of individual (Bower & Hilgard, 1981; Gergen, 1995). In this light, the representation of knowledge from only one perspective does not give a practical solution. On the one side, the problem of exogenic view of knowledge is how the external world is made manifest to the internal and how the subjectivities can ever record or ascertain the nature of the so-called objective world (Bower & Hilgard, 1981; Gergen, 1995). On the other side, the problem of endogenic view of knowledge is to understand or comprehend the subjectivities of the others and to ascertain whether the externalised source is the reflection of internal state (Bower & Hilgard, 1981).

The construction and representation of knowledge according to Gergen (1995) is carried out by "social constructionist orientation of knowledge" (p. 23) in which social interchange has a major role in constructing and representing knowledge. Explicitly speaking, the construction and representation of meaning is achieved through social interdependence which is context dependent and that serves communal functions (pp. 24-26).

Looking at the radical constructivism and social constructionism, we can notice that there is a difference between two theories in terms of representation of knowledge. The radical constructivism focuses on internal representation and social constructionism focuses on shared representation. The following figure can summarize the views of constructivism and constructionism regarding representation.

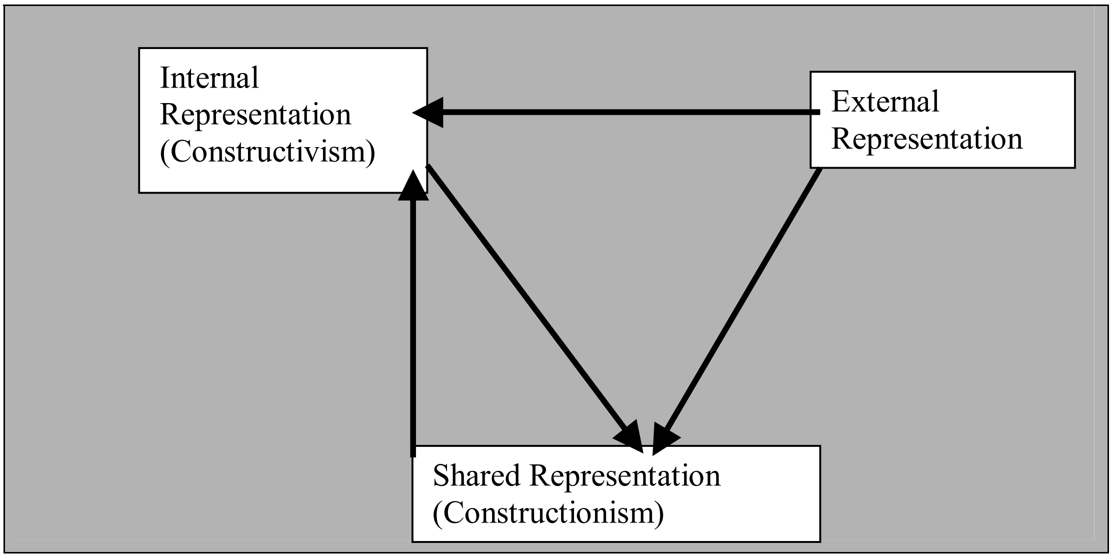

Figure 1. Representation and learning process according to Constructivism and constructionism

Vygotsky, Piaget and Representation

Vygotsky argued that advanced concepts appear first in social interaction, and only gradually become accessible to an individual (Confrey, 1995). Vygotsky was influenced by Marxist view which explicates that knowledge is constructed as a consequence of pre-existed matter or tools (Confrey, 1995). Moreover, Vygotsky argued for the evolution of higher cognitive process from social to individual (Confrey, 1995). In other words, the knowledge is "external" (p.189) in the beginning, which is, "eternalised" (p.189) later. Here, Vygotsky clarifies that internalisation is not the "transferal" (p.189) of an external activity but it is a process of gaining control over external "sign" forms (pp.189-190).

Piaget mentioned a different representational system from the representational system of Vygotsky. The focus of Piaget is on the subjectivity of representation, and the process of internalisation, according to him, is performed through interaction with the physical reality (Confrey, 1995). Furthermore, the internalisation according to Piaget is "schemata that reflect the regularities of an individual's physical action" (p. 200). On the contrary, Vygotsky thinks the internalisation as a social process. Moreover the representational system in Vygotskian perspective is more shared and external in the beginning and internalised later. The following figure can help visualise the both perspectives.

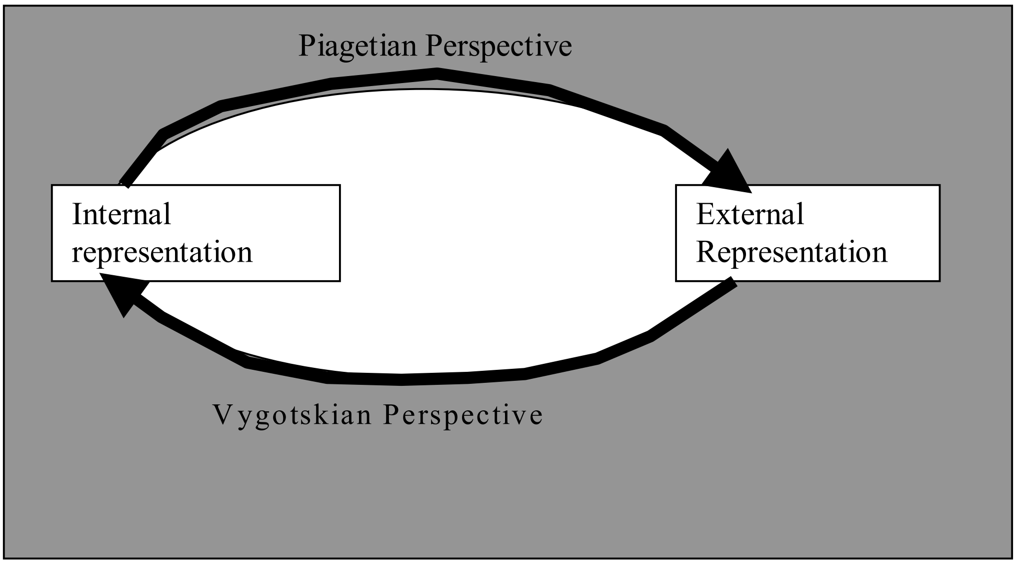

Figure 2. Representation According to Piagetian and Vigotskian Perspective

Ethno-mathematics and Representation

The term ethno-mathematics is used to express the relationship between culture and mathematics (D'Ambrosio, 2001). It is a new idea of studying mathematical representation from different cultural perspective. Ethno-mathematics is concerned with the connections that exist between the symbol, the representation and imagery
(Vergani, 1998). Moreover, representation from ethnomathematical perspective has a wider scope since different cultures have different types of representational systems, which would be useful in mathematics learning.

It is apparent that the development of ethno-mathematics has tried to transform the traditional concept of Euro-centred mono-representational system of mathematics to world centred multi-representational system of mathematics (D'Ambrosio, 2001). Ethno-mathematics does not study only the number system and symbols of different ethnic groups but also studies about the representational system of different aspect of their culture. The representational systems of a culture depend upon the types of mathematical knowledge that it has. For instance, some cultures have logico-mathematical knowledge (Kamii et al., 2001); some deals with narrative knowledge(Seeger, 1998); and some culture deal with paradigmatic knowledge (Seeger, 1998).

The tradition of representation started from the ancient civilisations. The representational system of early Babylonian was more mathematical while comparing with the Egyptians (Eves, 1969). The Mayan represented the number as a positional base-twenty system (Lara-Alecio, Irby, & Morish-Aldana, 1998). The Chinese represented multiplicative number system (Eves, 1969) while the Indian and Arabic represented the place value systems. The cultural artefacts, language, myths and literature help pinpoint the representational system of different cultures and civilisation.

Ethno-mathematics has an important role in learning mathematics hence in representational system. Particularly, it is important to contextualise the representational system. The case of Maori based mathematics teaching (Aspin, 1995) is an evident example in this regard.

Problem Solving and Representation

Problem solving creates a context for mathematics learning (Schoenfeld, 1992) and representation of mathematical ideas (Verschaffel & Corte, 1997). In the transition of learning from concrete-operational to formal-operational stage (Piaget & Inhelder, cited in Schoenfeld, 1992), the child learns in the continuum of interaction with the concrete/semi-concrete object to the situation which embeds those concepts with what they learned from the milieu (Schoenfeld, 1992; Verschaffel & Corte, 1997). Moreover, the situation that is transformed from the representation of concepts through concrete objects to virtual situation is problem-solving situation.

Problem solving is one of the dominating learning activities featured throughout the history of mathematics (Eves, 1967) for developing each new mathematical concept and their representational systems. Furthermore, it plays a vital role in developing the representation of mathematics learning.

According to Stanic and Kilpatric (cited in Schoenfeld, 1992) there are five roles of the problem solving activities in mathematics. They can be listed as follows:

- As a justification for teaching mathematics
- As a motivating factor to learn mathematical contents
- As recreation
- As a means of developing new skills
- As practice

The role of problem solving according to Polya (1945) is viewed as *instructional context* (cited in Schoenfeld, 1992). Epistemological link of this view is with the absorptive and transmitive function of problem solving, as it was a dominant notion of learning before mid 80s (cited in Schoenfeld, 1992). As the constructivism became a dominant model of leaning process, the role of problem solving has changed from transmitive to constructive role because of its intertwined relation with construction and representation of mathematical ideas (Verschaffel & Corte, 1997; vonGlasersfeld, 1995; Voutsina & Jones, 2001).

The role of word problems is vital in early primary grades (Verschaffel & Corte, 1997) as they are very useful to apply and represent their thinking and understanding of mathematical ideas . The focus of research in problem solving has changed from result-based to problem-based direction (Verschaffel & Corte, 1997). Furthermore, the recent trend of research in this area is on multiple representations of problems rather than calculating the total number of students who were able to find the correct solution (Lowrie, 2001; Verschaffel & Corte, 1997; Boulton-Lewis & Tait, 1993).

It is also important to discuss about the different types of situation of addition and subtraction-related (word) problems. According to Fuson (1992), three general additive and subtractive situations have been suggested in order to represent the problem-solving situation of addition and subtraction. The following table helps visualize the real world representation of addition and subtraction-related problems:

Table 1

Additive and subtractive situation

S. N.	General situation	Additive	Subtractive situation
1.	Active situation Unary operation ($Q_a \rightarrow Q_b$)	Start +Change = End (Change add to)	Start – Change = End (Change take from)
2	Active situation Binary operation (Q_1, Q_2) $\rightarrow Q_3$	Part + Part =All (Combine Physically)	Equalise (Change take from and change add to)
3.	Static situation Binary operation (Q_1, Q_2) $\rightarrow Q_3$	Part + Part =All (Combine Conceptually)	Compare (How many more and how many less?)

Source: (Fuson, 1992)

Basically, there are four addition and subtraction situations that represent the real world: Compare, Combine, Change add to and Change take from (for detail see, Fuson, 1992). Such four situations spread over the three additive and subtractive situation as categorised as active-unary, active-binary and static-binary operational situations.

The classification of word problem of Vergnaud (1982) is dissimilar with that of the Fuson (1992) (as cited in Verschaffel & De Corte, 1997). Vergnaud's (1982) classification comprised of six categories to classify the word problems related to addition and subtraction. The classification is as follow:

- Composition of two measures
- Transformation linking two measures
- Static relationship linking two measures
- Composition of two different transformations
- Transformation linking two static relationships
- Composition of two static relationships

The first three categories are closer to the classification of Fuson (1992). However the other three categories are fundamentally different (Verschaffel & De Corte, 1997). Despite the differences in detail classification, the basic classification of addition and subtraction problem situations is almost same as proposed by Fuson (1992).

The "developmental levels of conceptual structures and solution procedure for word problems" (Fuson, 1992, pp. 252-253) consists of three major levels. Such levels starts from the "single representation of an addend or the sum" and attains the level of "derived fact and known fact procedures" (Fuson, 1992). The brief summary of those levels are presented in the following table:

Table 2

Developmental levels of conceptual structures for word problems

Levels	Counting and cardinal conceptual units	Cardinal conceptual operation
The single representation of addend or the sum	Perceptual unit items: single [re]presentation of the addend or the sum	Cardinal integration
Abbreviated sequence counting procedures	Sequence unit items: Simultaneous [re]presentation of each addend within the sum	Embedded integration for both addends
	Unproduced first addend sequence unit items	First addend is embedded and abbreviated by a cardinal-to-sequence transition
	Keeping-track unit items: Second addend entities are generated to correspond with the second addend words	Second addend: Paired integrations of the second addend and keeping-track unit items
Derived fact and known fact procedures	Ideal chunkable unit items: Simultaneous non-embedded mental [re]presentation of both addends and the sum	Numerical equivalence Non-embedded simultaneous addend and sum
	Numbers as unit	Triad addend/addend/sum structure

Source: *Fuson (1992)*

Fuson's (1992) levels and associated conceptual operations are quite useful in developing the level of representation. However, the use of the term "representation" and "presentation" to convey the same meaning is an epistemological incompleteness of Fuson's (1992) classification. It seems the classification is more useful in teaching rather than in learning.

According to Verschaffel and De Corte (1997), the problem-solving activity starts with the construction of network of syntactic representation of the relationship of the quantities involved in the problem. Further they suggest that the constructive process of representation of the problematic situation is the second stage. In this stage three kinds of knowledge are very important which are known as: schemata of problem situations, linguistic knowledge, and knowledge about the game of school word problems (Verschaffel & De Corte, 1997, p 76).

Verschaffel and De Corte's (1997) focus on constructing schemata of the problems resembles to the concepts of Piaget's (1977) schemata of representation of learning (cited in Confrey, 1995). The schemata in problem solving are the sets of plan, which are mentally constructed and represented by the learner (Verschaffel & De Corte, 1997).

Problem-solving process elicits both cognitive and metacognitve (Curcio & Artzt, 1998) activity in which the possible problem solving behaviour can be listed as follows (see Curcio & Artzt, 1998):

- Reading the problem (cognitive)
- Understanding the problem (metacognitive)
- Analysing the problem (metacognitive)
- Planning (metacognitive)

In the case of multiple of representation of addition and subtraction related problems, the metacognitive behaviour is vital. Furthermore, the metacognitive behaviour helps a establishing a control over the representational method and problem solving technique (Curcio & Artzt, 1998; Verschaffel & Corte, 1997) that probes the understanding of mathematical problems and problematic situation.

Research in Representation of Mathematical Concepts

Research in the field of representation has been carried out in order to identify the representation of mathematics learning for the last two decades. Such research has focused on student-generated representation of basic mathematical concepts and corresponding problem situations. In this connection, this section seeks to carry out a descriptive review of related research, which will be followed by a critical analysis of research that would be vital to develop a model for the proposed research especially to justify its rationale from ontological and epistemological point of view (Merrian, 1988) of the term “representation”.

Under a theoretical framework of three categories of problem-solving approaches such as, visualizers, verbelisers, and both user and the role of imagery in problem-solving, Lowrie (2001) reported that 42% of the participants solved the mathematical problems by using the visual techniques and 58% solved non-visually. The research was carried out in order to examine the student-generated representation of problems in visual and non-visual continuum and at the either end of the continuum in relation to the performance of mathematics, and to identify effectiveness of the approach of representational systems to solve the problem (Lowrie, 2001). The tools were MPI (Mathematical Processing Instrument) developed by Suwarsono (1982) (cited in Lowrie, 2001), which comprised of twenty mathematical problems and a corresponding questionnaire for generating information regarding the student-generated problem-solving techniques (Lowrie, 2001). In order to calculate the “visuality measure” (Lowrie, 2001, p. 356) and “preference efficiency” (Lowrie, 2001, p. 356), the responses were “marked in the range of +2 to –2” (Lowrie, 2001, p. 356). Statistically, there was no effect of visual-nonvisual methods upon the successes in solving the problem (Lowrie, 2001, p. 357).

A study carried out by Swafford and Langrall (2000), aimed at determining the use of equations for describing and representing the contextual problem solutions prior to the formal learning of algebra, reported that students were found to be able to solve the problems involving specific cases with the exception of “car wash problem” (Swafford & Langrall 2000, pp. 97, 32). Similarly, it was reported that majority of the students described the functional relationship between the variables rather than recursive ones. Those descriptions were found in the “refund problem”, the “wage problem”, and the “border problem”(Swafford & Langrall, 2000, p. 93). Regarding the task of symbolic representation, it was reported that all the students but one were able to generate an equation for at least one of the situations.

The theoretical background of Swafford and Langrall’s (2000) study is based on the perspective of historical development of algebra. Broadly speaking, the development of present mathematics is a gradual advancement of its representational system (Goldin & Shteingold, 2001) which is almost similar with the paradigm of learner-generated representational system (Swafford & Langrall, 2000). The tool of

Swafford and Langrall's (2000) study was a set of interview items in which six different types of problems were included, for example, "refund problem" (direct variation/proportionality), "hours and wage problem" (linear relationship), "border problem" (linear relationship, geometric concepts), "concert hall problem" (arithmetic sequence), "paper folding problem" (exponential) and "car wash problem" (inverse variation) (Swafford & Langrall, 2000, p. 93). The sources of data were transcribed interviews, interviewers' note, and students' written work (Swafford & Langrall, 2000).

Research on graphical representation of bivariate and multivariate data by grade 4,5 and 6 students revealed three levels of student-generated graphical representation of bivariate and multivariate association (Moritz, 2000). The research had been carried out as an extension of the previous studies as conducted by Mevarech and Kramasky (1997) and Moritz and Watson (2000)(cited in Moritz, 2000). Regarding the tools of Moritz's (2000) study, it was reported that a survey item was prepared including two parts, namely part (B): items related to bivariate association and part (M): items related to multivariate association. During the data collection procedure, one item was asked from the related area and the other three items were asked from unrelated ones. The levels of responses were categorized as level1: unsuccessful bivariate association, level 2: partial bivariate association, level 3: complete bivariate solution and the case of multivariate representation. The numerical (symbolic) analysis revealed that there was an association of the response level to part (M) according to the response level of part (B).

Diezmann (1999) reported that in order to develop the students' ability of using diagram as cognitive tools, teachers need to assess the quality of diagrams and provide them with the necessary support. The theoretical framework of Diezmann's (1999) research on "Assessing the diagram quality: Making difference to representation" has been built according to the previous research on representation of mathematical problems by using diagrams. Furthermore, matrices, networks and hierarchies and a range of diagrams that represent part-whole characteristics (Diezmann, Francis, Horley & Novick, cited in Diezmann, 1999) were taken into account of this study. The aim of Diezmann's (1999) study was to explore how the quality of diagrams can be assessed using theoretical prototypes, and specifically, how prototypes can be used to identify the different levels of performance.

Diezmann's (1999) research was a case study of 12-year-old five students who were both from high and low achiever in mathematics as well as high and low performers in visual methods of solution. The instruction of twelve half-hour lesson addressed the four general purposes of diagrams generation and its use in novel problem-solving tasks. Interviews were conducted before and after the instruction and along with the task related to the five "isomorphic problems" (Diezmann, 1999, p. 186). The levels of student-generated diagrams were identified on the basis of the following criteria: Level 0 was assigned when there was no diagram; level 1 was categorised for the plausible diagram but lacking on assigning the appropriate component to structure; level 2 was assigned for the diagram which represented at least one but not one component of the structure; and level 3 was labelled for such diagram that represented all the components of the structure appropriately.

Regarding the different levels of representation of student-generated matrix and part-whole diagram, it was reported that the categorisation of student-generated matrix diagram was easier than the student-generated part-whole diagram (Diezmann, 1999). The result would be applicable to other situation if more than one source of data were used to identify the level of student-generated representation of part-whole diagram.

A research on representation of numerical (symbolic) situation through problem-solving at kindergarten (KG) level, carried out by Outhred and Sardelich (1997) concluded that students' representation of numerical (symbolic) situation was found better in the second time than in the first one. Moreover, it had been concluded that students understood that a cube represents unit for representing a number. Similarly, it was reported that students developed the ability of writing equation in the end of five-month problem-solving sessions. The research was a result of those data revealed through a regular KG classroom. A problem sheet was prepared comprising of "addition" (combine), "subtraction" (combine and separate), "multiplication" (equal groups), "division" (partitive) and "fraction" (one half) (Outhred & Sardelich, 1997, p. 378) related problems.

The data collection procedure was unanticipated as they were collected during the problem-solving session focusing on student-generated problem-solving strategies, for example, "crossing out and partitioning sets for subtraction", "separation for subtraction and addition", "drawing lines for sharing relationships between solving strategies" and "letters and words to label elements of sets or set"

(Outhred & Sardelich, 1997, p. 380). The student-generated drawings, produced during the problem-solving session, were kept as a basis for analysis.

Outhred & Sardelich's (1997) research was an extension of the previous research carried out by Carpenter et. al (1993) as they concluded that kindergarten students had solved a variety of difficult problems in the end of eight month's problem-solving session (Outhred & Saradelich, 1997).

As the data was collected only from a classroom, there is a question of representativeness and generelizability. On the contrary, as a qualitative research, Outhred & Sardelich's (1997) study could contribute to the development of further research in the area of representation of numerical (symbolic) problems provided the triangulation measures were used in data collection.

The effect of "consistent language (CL)" (Verschaffel, 1994, p. 146) and "inconsistent language (IL)" (Verschaffel, 1994, p. 146) in problem-solving process was investigated by Lewis and Mayer (1982) in reference to the "compare problems" (Verschaffel, 1994, p. 147), which revealed that the problem solvers could make more "reversal errors" (Verschaffel, 1994, p. 148) on IL rather than CL problems. The findings of the research carried out by Lewis and Mayer (1982, cited in Verschaffel, 1994) were hypothesized by Verschaffel (Verschaffel, 1994, p. 146) in order to carry out a research on "Using retelling data to study elementary school children's representations and solutions of compare problem" (Verschaffel, 1994, p. 143). Forty 11-year-old students were selected for the study. For the preparation of data collection tool, an analysis of the arithmetic textbook was carried out which revealed that the textbooks contained fewer compare problems (Verschaffel, 1994). Similarly, interview with the teachers was carried out to ensure whether or not they had explicitly taught to solve IL problems by transforming to CL problems.

The tool used in the study was a "problem card" consisting nine items comprising of "buffer items" and "target items" (Verschaffel, 1994, p. 143). Students had to solve each problem and later they had to retell the same problem by reading two numbers mentioned on the other side of the card.

The findings were reported explicitly as the CL problems elicited more correct arithmetic operations rather than the IL problems did. Similarly, the students solved the CL problems faster than the IL problems. As concerned to the retelling protocols, the higher frequency was found in CL problems than in the IL problems. Furthermore, only one "C/I inversions" (Verschaffel, 1994, p. 153) was found on CL problems and at least 61% C/I inversion was found on IL problems.

The research developed its theoretical framework from the previous research on mathematical problem with consistent language (CL) and inconsistent language (IL) (Verschaffel, 1994). However, there was no explicit discussion about the different types of representational systems and their role in problem solving. Regarding the problems with CL and IL, it could be discussed in relation to the representational systems, which could play a vital role in developing understanding of mathematics learning. Moreover, the process of "retelling" (Verschaffel, 1994, pp. 141, 143) was focused on memorization rather than construction of the problems. Similarly, it would be better if the study were based on the multiple sources of data.

Boulton-Lewis and Tait (1993) aimed at searching the student-generated process of representing addition operation associating with "analogs" (Boulton-Lewis & Tait, 1993) and the use of analogs to solve the given problems (Boulton-Lewis & Tait, 1993). The research was carried out under a theoretical framework of cognitive theories focusing on paradigmatic representational systems according to which representation of learning starts from lower and attains to higher level (Boulton-Lewis & Tait, 1993). Furthermore, the theoretical framework of Boulton-Lewis and Tait's (1993) research has discussed about the findings of previous research on concrete representation as quoting concrete representation of a mathematical concept could be more effective for young children to understand the representation of mathematical concepts (Boulton-Lewis & Tait, 1993).

The tool of Boulton-Lewis and Tait's (1993) study comprised of "two-addenda problem set" (Boulton-Lewis & Tait, 1993, pp. 131-132) in which the problems comprised a mix of one-, two- and three-digit numerals. In order to generate the data effectively, the interviewer and students discussed about the materials (multibase arithmetic block, sticks singly, counters, unifix, symbols, problems written on the card, paper and pencil) and their use in solving the corresponding problems. Furthermore, the children were encouraged to talk about the process of problem solving. The result indicated that children preferred to use verbal and mental strategies rather than formal algorithms: they did not want to use analogs unless they could not perform the task in any other way.

The major finding of the study was: "Children preferred to use verbal and mental strategies rather than formal algorithms and did not want to use analogs unless they could not perform the task any other way" (Boulton-Lewis & Tait, 1993, p. 133). The research could discuss about the different types of analogs and their use in representation of problem-solving strategies. Similarly, the sought category of the problem-solving strategies such as material, verbal, mental and mixed could be discussed precisely.

Critical Analysis of the Research in the Area of Representation

This section deals with a critical analysis of the previous research and tries to locate the present research problem.

According to the review of the above-discussed research reports, it reveals that some aspects of the proposed research problem have been discussed by the previous research. However, they were not sufficient to address the student-generated multiple representation of addition and subtraction related problems. The different types of problem representation of sixth-grade students (Lowrie, 2001) and the study of development of nine KG children's representations of numerical (symbolic) problems (Outhred & Sardelich, 1997) were almost the same in terms of the framework and procedure. The previous was more triangulated than the later one in terms data collection procedure. However, there was a lack of a comprehensive theoretical framework for the representational systems suggested by different schools of learning theories. In relation to the methods of assessing the level of student-generated diagram, the study carried out by Diezmann (1999) is more contextual than the others though it lacks multiple sources of data in classifying the part-whole diagrams. Methodological triangulation was considered in "The use of retelling method in order to identify the elementary schoolchildren's representation and solution of compare problems" (Verschaffel, 1994, pp. 147-158) by using three different sources of data. However, this research lacks the theoretical triangulation for developing the ontological and epistemological background for the term "representation". Furthermore, the representation of day-to-day language in terms of graph and their levels as carried out by Moritz (2000) is an example of classification of student-generated graphs, which can be useful in developing tools in order to assess student-generated diagrams of addition and subtraction problems.

The need of multiple representations of problems and problem-situation has been realized since long ago as a means of developing and probing the understanding of mathematical concepts (Schoenfeld, 1992). Until recently, the research, carried out to identify the student-generated diagrammatic representation (Moritz, 2000; Diezmann, 1999; Outhred & Sardelich, 1997), has focused on student drawing (Outhred & Sardelich, 1997) and sketching the graph of verbal problems (Moritz, 2000; Diezmann, 1999). The other type of representation was discussed as verbal representation by retelling the problem on the basis of given numbers (Verschaffel, 1994). Furthermore, the representation by equation and numerical (symbolic) solutions were also discussed and labelled as symbolic representation (Swafford & Langrall, 2000; Boulton-Lewis & Tait, 1993). However, there is still a need of research, which deals three major types of representational systems (verbal, diagrammatic and numerical (symbolic)) involved in mathematics learning.

The notion of representation according to the above-mentioned research is construction, abstraction and demonstration of mathematical problems (Moritz, 2000; Diezmann, 1999; Outhred & Saradelich, 1997; Verschaffel, 1994; Swafford & Langrall, 2000; Boulton-Lewis & Tait, 1993). Multiple representations, according to Schoenfeld (1992), are essential while solving the mathematical problems. Specifically, the addition and subtraction problems are represented in three ways: unary operation (change), binary operation (combine physically), static situation (combine conceptually) (Fuson, 1992). This classification is essential for developing the tool of the proposed study.

Generally, theoretical representation of addition and subtraction-related problems are carried out by the same representational technique (Fuson, 1992). However, there was a difference in representational system regarding the different types of problems related to addition and subtraction (Boulton-Lewis & Tait, 1993). On the ground of this contradiction, it is essential to investigate whether there would be difference between the representational systems of addition and subtraction-related problems.

Diagrams are supposed to be one of the means of representation of mathematical problems, as “diagrammatic representation” (Diezmann, 1999, p. 185-190) has become a very popular research area in recent days. However there is a lack of research on comparison between diagrammatic and other types of representational systems. The present research will be focused on a comparison between student-generated diagrammatic verbal and numerical (symbolic) representation.

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