Sharply transitive permutation groups

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This paper started from the fact that Möbius transformations are sharply 3-transitive, and a vague recollection that there are sporadic simple groups which are multiply transitive (the Mathieu groups). Most - if not all - of the contents is not new, I think this is 19th century math. It was a nice exercise though, and got me playing with permutations, which is a lot of fun.

1 Definitions

Let S_n be the permutation group, acting on $X = \{1, 2, \ldots, n\}$. Let $G \subset S_n$ be a subgroup.

Definition 1.1 *G is called k-transitive if for every k non repeating elements $x_1, x_2, \ldots, x_k \in X$ and every k non repeating elements $y_1, y_2, \ldots, y_k \in X$, there is a permutation $g \in G$ such that $g(x_i) = y_i$ for all $i = 1, \ldots, k$.*

Proposition 1.2 *G is k-transitive if and only if for every k non repeating elements $y_1, y_2, \ldots, y_k \in X$ there is a permutation $g \in G$ such that $g(i) = y_i$ for all $i = 1, \ldots, k$.*

Proof: Assume G is k-transitive. Let $y_1, y_2, \ldots, y_k \in X$ be k non repeating elements. Choose $x_i = i$ for $i = 1, \ldots, k$. Using definition 1.1 there exists a permutation $g \in G$ such that $g(i) = g(x_i) = y_i$ for all $i = 1, \ldots, k$.

Assume the existence condition in the proposition. Let $x_1, x_2, \ldots, x_k \in X$ and $y_1, y_2, \ldots, y_k \in X$ be two sets of non repeating elements. By our assumption there exist two permutations $g_1, g_2 \in G$ such that $g_1(i) = x_i$ and $g_2(i) = y_i$ for all $i = 1, \ldots, k$. The permutation $g = g_2 g_1^{-1} \in G$ satisfies $g(x_i) = g_2 g_1^{-1}(x_i) = g_2(i) = y_i$.

■

Definition 1.3 *G is called sharply k-transitive if G is k-transitive, and for every k non repeating elements $x_1, x_2, \ldots, x_k \in X$ if $g_1, g_2 \in G$ satisfy $g_1(x_i) = g_2(x_i)$ for all $i = 1, \ldots, k$, then $g_1 = g_2$.*

Proposition 1.4 *G is sharply k-transitive if and only if for every k non repeating elements $y_1, y_2, \ldots, y_k \in X$ there exists a unique permutation $g \in G$ such that $g(i) = y_i$ for all $i = 1, \ldots, k$.*

Proof: Assume G is sharply k-transitive. Let $y_1, y_2, \ldots, y_k \in X$ be k non repeating elements. Using proposition 1.2 there exists a permutation $g \in G$ such that $g(i) = y_i$ for all $i = 1, \ldots, k$. If $g' \in G$ also satisfies this condition, then using definition 1.3 we have $g = g'$, so g is unique.

Assume the existence condition in the proposition. By proposition 1.2 G is k-transitive. Let $x_1, x_2, \ldots, x_k \in X$ be a set of non repeating elements. Suppose $g_1, g_2 \in G$ satisfy $g_1(x_i) = g_2(x_i)$ for all $i = 1, \ldots, k$. Applying g_2^{-1} to both sides we get $g_2^{-1} g_1(x_i) = x_i$ for all $i = 1, \ldots, k$. By k-transitivity there exist a permutation $g \in G$ such that $g(i) = x_i$ for all $i = 1, \ldots, k$. The permutation $h = g^{-1} g_2^{-1} g_1 g \in G$ satisfies

$$h(i) = g^{-1} g_2^{-1} g_1 g(i) = g^{-1} g_2^{-1} g_1(x_i) = g^{-1}(x_i) = i.$$

The identity permutation $e \in G$ also satisfies $e(i) = i$ for all i. By our uniquness assumption $e = h = g^{-1} g_2^{-1} g_1 g$, so $g_1 = g_2$. ■

2 Basic properties

Proposition 2.1 *If $G \subset S_n$ is sharply k-transitive then $|G| = \frac{n!}{(n-k)!}$*

Proof: By proposition 1.4 every choice of k non repeating elements $y_1, y_2, \ldots, y_k \in X$ defines a unique permutation $g \in G$ such that $g(i) = y_i$. The size of G is just the number of these choices, which is $\frac{n!}{(n-k)!}$. ■

Proposition 2.2 *S_n is sharply n-transitive and sharply $(n-1)$-transitive.*

Proof: Every choice of n non repeating elements $y_1, y_2, \ldots, y_n \in X$ defines a unique permutation $g \in S_n$ such that $g(i) = y_i$ for all $i = 1, \ldots, n$. This shows that S_n is sharply n-transitive, using proposition 1.4.

For every choice of $n-1$ non repeating elements $y_1, y_2, \ldots, y_{n-1} \in X$ we can define $g(i) = y_i$ for all $i = 1, \ldots, n-1$. There exists a unique value $y \in X$

not in $\{y_1, \ldots, y_{n-1}\}$, and $g(n) = y$ is the unique extension to a permutation $g \in S_n$. Again using proposition 1.4 S_n is sharply $n-1$-transitive. ■

Proposition 2.3 *$A_n \subset S_n$ is sharply $(n-2)$-transitive.*

Proof: For every choice of $n-2$ non repeating elements $y_1, y_2, \ldots, y_{n-2} \in X$ we can define $g(i) = y_i$ for all $i = 1, \ldots, n-2$. There exist two values $a, b \in X$ not in $\{y_1, \ldots, y_{n-2}\}$, and there are two possible extensions to a permutation: $g_1(n-1) = a$ and $g_1(n) = b$ or $g_2(n-1) = b$ and $g_2(n) = a$. Since $g_1 = g_2(n-1\,n)$, exactly one of these is an even permutation. and this is the unique extension to a permutation $g \in A_n$. Using proposition 1.4 A_n is sharply $n-2$-transitive. ■

Proposition 2.4 *let $G \subset S_n$ be sharply k-transitive, and let $g \in G$.*

1. *If g stabilizes k elements $x_1, \ldots, x_k \in X$ then g is the identity permutation.*
2. *Suppose $g \in G$ has n_l cycles of length l with $n_l l \geq k$. Then g is of order l and all cycle lengths are divisors of l.*

Proof:

1. The identity permutation stabilizes all elements in X. By definition 1.3 g is the unique permutation in G that stabilizes $x_1, \ldots, x_k$, so g is the identity.
2. The permutation g^l stabilizes all elements in X that are in cycles of length dividing l. Since there are at least $n_l l \geq k$ of these, g^l must be the identity by part 1, and all cycle lengths of g divide l. The order of g is then exactly l since g contains cycles of length l.

■

Proposition 2.5 *Let $G \subset S_n$ be sharply k-transitive with $k > 1$, and let $H \subset G$ be the subgroup stabilizing n. Then, restricting to $1, \ldots, n-1$, H is a sharply $(k-1)$-transitive subgroup of S_{n-1}.*

Proof: By definition 1.3, for every $k-1$ non repeating elements $y_1, \ldots, y_{k-1} \in X$ different from n there exists a unique $g \in G$ such that $g(i) = y_i$ for all $i = 1, \ldots, k-1$, and $g(n) = n$. g is in H since g stabilizes n. Restricting to $1, \ldots, n-1$ proposition 1.4 shows that $H \subset S_{n-1}$ is sharply $(k-1)$-transitive. ■

3 Sharply 1-transitive groups

The simplest example of a sharply 1-transitive group in S_n is the cyclic group of order n generated by an n-cycle, say $(1\,2\,\ldots\,n)$. The smallest example of a sharply 1-transitive group which is not cyclic, is the four group in A_4, consisting of $\{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. In fact, the construction used in he proof of Cauchy's theorem (that all finite groups are isomorphic to permutation groups) provides more examples of sharply 1-transitive groups.

Let G be a finite group of order n with elements $x_1, x_2, \ldots, x_n$. For any element x_i define the left multiplication operator L_{x_i} by $L_{x_i}(x_j) = x_i x_j$. L_{x_i} is 1-1 since $L_{x_i}(x_j) = L_{x_i}(x_k)$ implies $x_i x_j = x_i x_k$ and then $x_j = x_k$. L_{x_i} is also onto since $L_{x_i}(x_i^{-1} x_j) = x_i x_i^{-1} x_j = x_j$ (a finiteness argument can also be applied). This shows that L_{x_i} are permutations in S_n.

The map $x_i \mapsto L_{x_i}$ is a group homomorphism $G \to S_n$ since $L_{x_i} L_{x_j}(a) = x_i x_j a = L_{x_i x_j}(a)$. This homomorphism is injective, since if $L_{x_i} = L_{x_j}$ then when applying on e we get $x_i = L_{x_i}(e) = L_{x_j}(e) = x_j$. This shows that G is isomorphic to a subgroup of order n in S_n. Moreover, this image is sharply 1-transitive: using proposition 1.4, since $L_{x_i}(e) = x_i$ the permutation is determined once the image of e is known.

We can also show that all sharply 1-transitive groups are the result of the above construction. Suppose $G \subset S_n$ is sharply 1-transitive. Let x_i be the unique element in G such that $x_i(1) = i$. Using left multiplication as above we get $L_{x_i}(x_j) = x_i x_j$, and the index is determined by $x_i x_j(1) = x_i(j)$. Thus $x_i = L_{x_i}$ as elements of S_n under this construction.

4 Sharply 2- and 3-transitive groups

As mentioned in the beginning, it is well known that Möbius transformations over finite fields are examples of sharply 3-transitive groups. We describe this example and use it to find sharply 2-transitive groups.

4.1 Möbius transformations

Let q be a prime power, F a field of size q. The group $\mathrm{GL}_2(F)$ of invertible 2 by 2 matrices over F acts on $F \cup \infty$ by

$$g(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d}$$

These are the Möbius transformations. If $c = 0$ then $g(\infty) = \infty$, otherwise $g(\infty) = \frac{a}{c}$ and $g(-\frac{d}{c}) = \infty$. It is well known that this defines a group action.

Proposition 4.1 *This group action is 3-transitive.*

Proof: Let $\{\alpha, \beta, \gamma\}$ be any three different element of $F \cup \infty$. The matrix $g_1 = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $g_1(\infty) = \alpha$, so $g_1^{-1}\{\alpha, \beta, \gamma\} = \{\infty, \beta', \gamma'\}$. If $\alpha = \infty$ we use the identity for g_1. The matrix $g_2 = \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix}$ satisfies $g_2(\infty) = \infty$ and $g_2(0) = \beta'$, so $g_2^{-1}g_1^{-1}\{\alpha, \beta, \gamma\} = \{\infty, 0, \gamma''\}$. Note that both β' and γ'' are not ∞, and both can be recovered explicitly from α, β, γ. The matrix $g_3 = \begin{pmatrix} \gamma'' & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $g_3(\infty) = \infty$, $g_3(0) = 0$ and $g_3(1) = \gamma''$, so $g_3^{-1}g_2^{-1}g_1^{-1}\{\alpha, \beta, \gamma\} = \{\infty, 0, 1\}$. The inverse element $g_1g_2g_3$ shows that the action is 3-transitive (proposition 1.2). ■

Proposition 4.2 *The only Möbius transformation that stabilizes 0, 1 and ∞ is the identity.*

Proof: Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such a transformation. From $g(\infty) = \infty$ we have $c = 0$, and from $g(0) = 0$ we have $b = 0$. The condition $g(1) = 1$ now implies $a = d$ and g is a scalar matrix. The transformation defined by g is then $g(z) = \frac{az}{a} = z$ so g is the identity. ■

The scalar matrices define the identity transformation, so the group of Möbius transformations is actually $\mathrm{PGL}_2(F)$.

Proposition 4.3 *The action of* $\mathrm{PGL}_2(F)$ *on* $F \cup \infty$ *is sharply 3-transitive.*

Proof: Suppose g_1 and g_2 are two Möbius transformations that satisfy $g_i(\infty) = y_1, g_i(0) = y_2, g_i(1) = y_3$. The transformation $g_1^{-1}g_2$ then stabilizes 0, 1 and ∞. Proposition 4.2 shows that $g_1 = g_2$. This uniqueness, together with 3-transitivity (proposition 4.1), completes the proof by proposition 1.4. ■

This section shows that there is an infinite family of sharply 3-transitive groups, acting on $q+1$ elements where q is a prime power. Except for a few small cases all these groups are simple (NO THEY'RE NOT !!, see note in the question section).

- For $q = 2$ this group is sharply 3-transitive in S_3 so it is isomorphic to S_3.
- For $q = 3$ this group is sharply 3-transitive in S_4 so it is isomorphic to S_4.
- For $q = 4$ this group is sharply 3-transitive in S_5 so it is isomorphic to A_5. It is also simple.

4.2 Affine groups

Using the group of Möbius tranformations and proposition 2.5, the $\mathrm{PGL}_2(F)$ subgroup stabilizing ∞ is sharply 2-transitive. Matrices stabilizing ∞ satisfy $c = 0$, these are the affine transformations $g(x) = ax + b$. This shows an infinite family of sharply 2-transitive groups, acting on q elements where q is a prime power. All these groups are not simple, since the subgroup of translations $g(x) = x + b$ is normal.

4.3 Questions

I raise the following questions as a reminder to myself, what more I should try to understand. These are not open questions, I'm sure at least most of the answers are known to the experts of the field.

- Are there other sharply 2-transitive groups?
- Are there other sharply 3-transitive groups?
- Can you prove $\mathrm{PGL}_2(F)$ is simple directly from transitivity? Does this extend to all multiply transitive groups? (Answer: NO. In fact, I was wrong here, since $\mathrm{PSL}_2(F)$ is simple and is normal of index 2 in $\mathrm{PGL}_2(F)$ for odd characteristic)

5 Sharply k-transitive groups, $k \geq 4$

We saw in section 2 that S_n and A_n are infinite families of sharply transitive groups of degree $n, n-1$ and $n-2$. In this section we will show that there are almost no other sharply k-transitive groups with $k \geq 4$.

Let $G \subset S_n$ be a sharply 4-transitive group. We assume $n \geq 6$, excluding S_4 and S_5. We will now prove that n is either 6 or 11. The idea is to study the elements of order 2 in G.

A permutation of order 2 has cycles of order 2 (at least one), cycles of order 1 (stationary points), and no other cycles. We would like to determine the exact cycle structure of an element of order 2 in G.

Proposition 5.1 *There is a unique element $g_0 \in G$ containing the cycles $(1\,2)(3)(4)$, g_0 is an element of order 2. If n is even then g_0 has exactly 2 stationary points and $\frac{n-2}{2}$ cycles of order 2. If n is odd then g_0 has exactly 3 stationary points and $\frac{n-3}{2}$ cycles of order 2. Moreover, all elements of order 2 in G have the same cycle structure as g_0.*

Proof: The cycles $(1\,2)(3)(4)$ are equivalent to the condition that the ordered 4-tuple $(1,2,3,4)$ is mapped to the ordered 4-tuple $(2,1,3,4)$. By definitions 1.1 and 1.3 there is a unique $g_0 \in G$ satisfying this condition. The element g_0^2 contains the cycles $(1)(2)(3)(4)$, mapping the ordered 4-tuple $(1,2,3,4)$ to itself, same as the identity element. By definition 1.3 g_0^2 is the identity, so g_0 is of order 2. The same argument shows that g_0 can have at most 3 stationary points, since it is obviously not the identity. The definition of g_0 includes two stationary points, so the number is 2 or 3. The number of stationary points is even iff n is even, since all other cycles are of order 2. Therefore for even n there are 2 stationary points, and for odd n there are 3. The number of cycles of order 2 follows immediately.

Suppose that $g \in G$ is any element of order 2. Again g has at most 3 stationary points, and at least 2 cycles of order 2, since $n \geq 6$. We choose two such cycles $(a\,b)(c\,d)$. The element g maps the ordered 4-tuple (a,b,c,d) to the ordered 4-tuple (b,a,d,c), and by definitions 1.1 and 1.3 only g has this pair of cycles of order 2. g_0 also has at least two cycles of order 2, choose $(i\,j)$ other than $(1\,2)$. By definition 1.1 there exists $h \in G$ mapping the ordered 4-tuple (a,b,c,d) to the ordered 4-tuple $(1,2,i,j)$. The elements g and $h^{-1}g_0h$ are identical on (a,b,c,d), so they are the same. This shows that g and g_0 are conjugate in G, and their cycle structure is identical. ∎

Proposition 5.2 *For any a, b, c, d there is exactly one element in G with the cycles $(a\,b)(c)(d)$, and exactly one element with the cycles $(a\,b)(c\,d)$.*

Proof: The ordered 4-tuple (a, b, c, d) is mapped to (b, a, c, d) in the first case, and to (b, a, d, c) in the second case. By definitions 1.1 and 1.3 there are unique elements in G satisfying these conditions. ■

We now count the number of elements of order 2 in G using the two cycle strutures of proposition 5.2.

We start with the structure $(a\,b)(c)(d)$. There are $\frac{n(n-1)}{2} \times \frac{(n-2)(n-3)}{2}$ ways to choose one 2-cycle and two 1-cycles, each choice defines a unique element of order 2 in G. There are many choices that lead to the same element in G, we count these using proposition 5.1. For even n the two 1-cycles are fixed but there are $\frac{n-2}{2}$ choices of 2-cycles that define the same element in G. For odd n there are 3 choices of two 1-cycles out of a given three, and $\frac{n-3}{2}$ choices of a 2-cycle. Therefore there are $\frac{n(n-1)(n-3)}{2}$ elements of order 2 in G for even n and $\frac{n(n-1)(n-2)}{6}$ for odd n.

Now we use the structure $(a\,b)(c\,d)$. There are $\frac{1}{2} \times \frac{n(n-1)}{2} \times \frac{(n-2)(n-3)}{2}$ ways to choose two 2-cycles, each choice defines a unique element of order 2 in G. We count multiplicities as before. For even n there are $\frac{1}{2}\frac{n-2}{2}(\frac{n-2}{2} - 1)$ choices of two 2-cycles that define the same element in G. For odd n there are $\frac{1}{2}\frac{n-3}{2}(\frac{n-3}{2} - 1)$ choices of a two 2-cycles. Therefore there are $\frac{n(n-1)(n-3)}{n-4}$ elements of order 2 in G for even n and $\frac{n(n-1)(n-2)}{n-5}$ for odd n.

Comparing the two ways of counting, we come to the conclusion that either $n = 6$ or $n = 11$. The unique solution for $n = 6$ is A_6. We will discuss the solution for $n = 11$ in the next section.

Using proposition 2.5 we find that for $k \geq 4$ sharply k-transitive groups can be found in S_{k+2} (these are exactly A_{k+2}) and in S_{k+7}.

Proposition 5.3 *There are no sharply k-transitive groups for $k \geq 6$ except A_{k+2}.*

Proof: We will prove that there are no 6-transitive groups in S_{13}, and using proposition 2.5 this completes the proof.

Assume G is a sharply 6-transitive group in S_{13}. Following the logic of proposition 5.1 every element of order 2 in G a has four 2-cycles and five 1-cycles. As above we count the number of order 2 elements in G using the same argument as proposition 5.2. We start with the cycle structure $(a\,b)(c)(d)(e)(f)$. There are $\frac{13 \cdot 12}{2} \times \frac{11 \cdot 10 \cdot 9 \cdot 8}{24}$ possible choices, and each element is chosen $4 \cdot 5$ times (one of four 2-cycles and four of five 1-cycles). The number of elements of order 2 is then $\frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{48\cdot 20}$. Now we use the structure $(a\,b)(c\,d)(e\,f)$. There are $\frac{1}{6}\frac{13\cdot 12}{2} \times \frac{11\cdot 10}{2} \times \frac{9\cdot 8}{2}$ possible choices, and each element is chosen 4 times (three of four 2-cycles). The number of elements of order 2 is then $\frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{48\cdot 4}$. The numbers are different, proving that there is no such G. ■

The conclusion is that for $k \geq 4$ the only sharply k-transitive groups are A_{k+2}, with the possible exception of a sharply 4-transitive group in S_{11} and a sharply 5-transitive group in S_{12}. These extra groups do exist, namely the Mathieu groups M_{11} and M_{12}.

6 The Mathieu M_{11} group

In this section we construct a sharply 4-transitive group in S_{11}, namely the Mathieu M_{11} group. The focus is on elements of order 2 as before. We will use the following proposition:

Proposition 6.1 *Let g and h be two commuting permutations of order 2. If $(a\,b)$ is a 2-cycle of g then exactly one of the following is true:*

- *h contains the cycles $(a)(b)$, or*
- *h contains the cycle $(a\,b)$ or*
- *h contains the cycles $(a\,c)(b\,d)$, where g contains $(c\,d)$.*

Proof: If $h(a) = a$ then $gh(a) = g(a) = b$, and since $gh = hg$ we get $b = hg(a) = h(b)$. This is the first case.

If $h(a) = b$ then $gh(a) = g(b) = a$, and similarly $a = hg(a) = h(b)$, this is the second case.

The remaining possibility is $h(a) = c$, where c is not a nor b. If we write $g(c) = x$ and $h(b) = y$ then $gh(a) = g(c) = x$ and $hg(a) = h(b) = y$. Thus $x = y = d$, which can not be a, b or c. Since both g and h are of order 2 we get the cycles of the third case. ■

By proposition 5.1 every element of order 2 in M_{11} has four 2-cycles and three 1-cycles. Up to isomorphism we may assume that M_{11} contains $g_1 = (1\,2)(3\,4)(5\,6)(7\,8)$. By 4-transitivity there is a unique element $g_2 \in M_{11}$ containing the cycles $(1\,2)(3)(4)$. Both g_1g_2 and g_2g_1 are identical on $(1, 2, 3, 4)$ so g_1 and g_2 commute. They share a 2-cycle $(1\,2)$, and since they are not identical elements they can not share another 2-cycle (sharp

4-transitivity). Proposition 6.1 with the cycle $(5\,6)$, together with the fact that g_2 already has two 1-cycles, shows that up to isomorphism we can choose $g_2 = (1\,2)(5\,7)(6\,8)(9\,10)$. Then $g_1g_2 = (3\,4)(5\,8)(6\,7)(9\,10)$.

Let g_3 be the unique element in M_{11} containing the cycles $(1\,3)(2\,4)$. As before g_1 and g_3 commute. We use the same argument, looking at the cycle $(5\,6)$ and using Proposition 6.1. Here, since all combinations of a pair of 2-cycles covering $\{5, 6, 7, 8\}$ are already present, g_3 can have either the cycle $(5\,6)$ or $(7\,8)$. Both are equivalent under isomorphism and we choose the first. g_2g_3 contains the cycle $(1\,3\,2\,4)$, so it is an element of order 4 by proposition 2.4. If the last 2-cycle of g_3 is not $(9\,10)$ then g_2g_3 also contains a 3-cycle, which is impossible. As a result, g_3 is $(1\,3)(2\,4)(5\,6)(9\,10)$. The group $< g_1, g_2, g_3 >$ is of order 8, including $g_1g_3 = (1\,4)(2\,3)(7\,8)(9\,10)$ and two more elements of order 4.

Let g_4 be the unique element in M_{11} containing the cycles $(1\,3)(2)(4)$. Now g_3 and g_4 commute, similar to the relation between g_1 and g_2. g_4 then must have two 2-cycles covering $\{5, 6, 9, 10\}$ and one 2-cycle from $\{7, 8, 11\}$. However, if 11 is in a 2-cycle then g_1g_4 has a 4-cycle covering $\{1, 2, 3, 4\}$ and a 3-cycle covering $\{7, 8, 11\}$, which is impossible. Since 9 and 10 are equivalent under isomorphism we can choose $g_4 = (1\,3)(5\,9)(6\,10)(7\,8)$. The group $< g_1, g_2, g_3, g_4 >$ is isomorphic to S_4, identical on 1-4 and embedded in 5-10. The remaining elements of order 2 in this group are $(2\,4)(5\,10)(6\,9)(7\,8)$, $(1\,4)(5\,6)(7\,9)(8\,10)$ and $(2\,3)(5\,6)(7\,10)(8\,9)$.

Let g_5 be the unique element of M_{11} containing the cycles $(7\,8)(9\,11)$. This element commutes with g_1, by proposition 6.1 the 1-cycles are 10 and one of the 2-cycles of g_1. Since all combinations of 2-cycles covering $\{1, 2, 3, 4\}$ are already present, there are four possible combinations: $(1\,5)(2\,6)$, $(1\,6)(2\,5)$, $(3\,5)(4\,6)$ and $(3\,6)(4\,5)$. The product of g_4 and $(1\,5)(2\,6)(7\,8)(9\,11)$ is $(1\,3\,5\,11\,9)(2\,6\,10)$, which is illegal. The same happens with the choice $(3\,5)(4\,6)$. Note that conjugating $< g_1, g_2, g_3, g_4 >$ with $(1\,3)(2\,4)(7\,8)$ results in $< g_1, g_1g_2, g_3, g_4 >$ which is the same. Therefore up to isomorphism we choose $g_5 = (1\,6)(2\,5)(7\,8)(9\,11)$.

We have constructed a group $M_{11} =< g_1, g_2, g_3, g_4, g_5 >$, and shown that it does not violate the conditions of sharp 4-transitivity, and that it is unique up to isomorphism. Using the GAP package we can show that the order of this group is $7920 = 11 \times 10 \times 9 \times 8$, and that it is in fact sharply 4-transitive. The 1, 2 and 3 point stabilizers can be computed. A small surprise is that the 1 point stabilizer of M_{11} is a non-simple sharp 3-transitive group. It contains a normal subgroup of index 2 which is isomorphic to A_6.