Reprinted from the Preprint Volume of the Ninth Symposium on Turbulence and Diffusion.
April 30-May 3, 1990, Roskilde, Denmark. Published by the American Meteorological Society, Boston, Mass.

P3.9

A CELL DYNAMICAL SYSTEM MODEL FOR TURBULENT SHEAR FLOWS IN THE PLANETARY ATMOSPHERIC BOUNDARY LAYER

A. Mary Selvam
Indian Institute of Tropical Meteorology, Pune 411008, India

1. INTRODUCTION

Long range spatial and temporal correlations in dynamical systems is identified as signatures of self-organized criticality (Bak, Tang and Wiesenfeld 1988) or deterministic chaos (Procaccia 1988). The strange attractor pattern characterising deterministic chaos in atmospheric flows is manifested as the fractal geometry to the global cloud cover pattern consistent with the inverse power-law form ν^{-B}, where ν is the frequency and B the exponent for the atmospheric eddy energy spectrum (Lovejoy and Schertzer (1986). The physical mechanism responsible for the observed robust spatiotemporal structure of the strange attractor of dynamical systems is not yet identified (Procaccia 1988). In this paper, a cell dynamical system model for turbulent shear flows in the ABL is presented.

2. CELL DYNAMICAL SYSTEM MODEL FOR DETERMINISTIC CHAOS IN DIGITAL COMPUTER REALIZATION OF NONLINEAR MATHEMATICAL MODELS.

2.1 Cell Dynamical System Model

In this nondeterministic computational technique, the dynamical system is assumed to consist of an assembly of identical unit cells. Starting with arbitrary initial conditions the evolution of the dynamical system proceeds at successive unit length steps during unit intervals of time following arbitrary laws of interaction between adjacent cells. The cellular automata belongs to the cell dynamical system described above and does not require calculus based long term integration schemes. However, the cellular automata rules for evolution do not have any physical basis. A cellular automata computational scheme incorporating the physics for model and real world atmospheric flows is described in the following.

2.2 Computer precision and deterministic chaos

Digital computer realizations of continuum mathematical models of dynamical systems are subject to the following inherent uncertainities characterizing deterministic chaos. (i) Sensitive dependence on initial conditions. (ii) Model results scale with computer precision. (iii) Model results exhibit periodicities related to computer precision (Beck and Roepstorff 1987). In the following it is shown that computer precision related roundoff error generates selfsimilar fractal internal structure to the phase space trajectory of the computed model output with sensitive dependence on initial conditions.

Let $OR_oR'_1$ (Fig.1) be a Euclidean straightline where $OR_o = R_oR'_1 = dR$. Let $OR_oR'_1$ be measured with a yardstick of length dR_o which is slightly less than dR. The measured length OR_1 is equal to $2dR_o$ and is slightly less than 2dR.

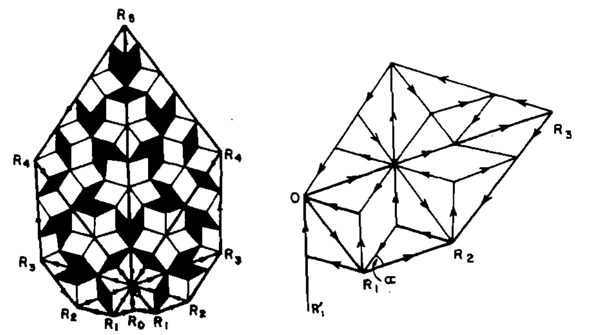

Figure 1 : The Internal structure of strange attractor design traced by digital computer realizations of nonlinear mathematical models of atmospheric flows and observed real world atmospheric flows.

The measurement procedure may be represented geometrically by the triangle OR_oR_1 where $OR_o=R_oR_1 = dR$. The original Euclidean straight line $OR_oR'_1$ now acquires a curvature and the apparent angular turning $d\theta$ caused by the imprecise measurement is $\widehat{R_oOR_1}$. In the ΔR_oOR_1 the apparent length OR_1 after two stages of imprecise measurement is equal to the vector sum of the first two length steps. All possible such apparent structure generated in the phase space by the computed output can be represented by a solid of revolution about the axis OR_o and whose vertical section is the triangle OR_1R_o. The spatial integration of the successive imprecise computational length step distributions give the spatial domain for the final computation W units of scale length R per unit. The discrete size of the yardstick generates apparent structure to the phase space domain of the computed output.

In the following the computational structure of numerical models is quantified by analogy with large eddy formations in turbulent shear flows. Townsend (1956) has shown that large eddies of appreciable intensity form as a chance configuration of the turbulent motion. Consider a large eddy of radius R which forms in a field of isotropic turbulence with turbulence length and velocity scales 2dR and w_* respectively. The dominant turbulent eddy radius is therefore equal to dR. The root mean square (r.m.s) velocity of circulation dW in the large eddy of radius R is

$$(dW)^2 = \frac{2}{\pi}\frac{dR}{R}w_*^2 \qquad (1)$$

The above analogy of the generation of coherent structures by small scale turbulent fluctuations in fluid flows is directly applicable to the generation of internal structure created by discrete yardstick length to the phase space trajectory of compted numerical model outputs. The similarity of deterministic chaos in numerical model outputs to turbulence in fluid flows was first pointed out by Ruelle and Takens (1971). Eq. (1) is directly applicable to digital computations of nonlinear mathematical models where w_* units of yardstick length dR result in a total computation of dW units of numerical length scale R. Eq.(1) gives the progressive increase of the number of units of precise computation w_* of scale length dR with increase in computational steps. Denoting by W_n and W_{n+1}, the successive n and (n+1)th intervals of computation with respective numerical length scales R_n = dR and R_{n+1}= R, Eq. (1) may be written as

$$W_{n+1}^2 = \frac{\pi}{2}\frac{R}{dR}W_n^2 \qquad (2)$$

The magnitude of the numerical length scale R_n(dR) increases with the computation. R_n is equal to W_n at the nth interval of computation beginning with unit values for R_n, R_{n+1} and W_n. The apparent curvature, i.e. angular turning dΘ in radians of the computed output W_{n+1} is given by R_n/R_{n+1}. The domain length R_{n+1} of the computed output is given by the cumulative vector sum of R_n, the yardstick length. The values of R_{n+1}, W_n, R_n dΘ, W_{n+1} and Θ are tabulated in Table 1.

Table 1 : The computed spatial growth of the strange attractor design traced by dynamical systems as shown in Fig.1.

R	W_n	dR	dΘ	W_{n+1}	Θ
1	1	1	1	1.254	1
2	1.254	1.254	0.627	1.985	1.627
3.254	1.985	1.985	0.610	3.186	2.237
5.239	3.186	3.186	0.608	5.121	2.845
8.425	5.121	5.121	0.608	8.234	3.453
13.546	8.234	8.234	0.608	13.239	4.061
21.780	13.239	13.239	0.608	21.286	4.669
35.019	21.286	21.286	0.608	34.225	5.277
56.305	34.225	34.225	0.608	55.029	5.885
90.530	55.029	55.029	0.608	88.479	6.493

It is seen that the phase space trajectory of the computed output W and the domain length R follow the Fibonacci mathematical number series. The internal structure of the phase space trajectory of nonlinear model outputs therefore consists of the quasiperiodic Penrose tiling pattern (Janssen 1988) as shown in Fig.1. It is seen from Table 1 and Fig.1 that starting from either side of the initial computation step OR_o the computation W proceeds in logarithmic spiral curves $OR_1R_2R_3R_4R_5$ such that one complete cycle is executed by the numerical computation W after 5 length steps of computation on either side of OR_o, i.e. clockwise and anticlockwise rotation. The overall envelope of the computation W follows the logarithmic spiral pattern. The incremental units of computation dW of domain length dR at any stage of computation is non-Euclidean because of internal structure generated by w_* units of discrete yardstick length dR. A measure of the departure from Euclidean shape of computed model output W caused by discrete yardstick length is derived as follows. Let k be the steady state fractional space (two-dimensional) occupied by the discrete yardstick length distribution in the two-dimensional Euclidean phase space of the computed output.

$$k = \frac{w_*}{dW}\frac{dR}{R} \qquad (3)$$

k is the non-dimensional steady state measure of the departure from Euclidean shape of the computed model output. Earlier it was shown that successive computational steps generate angular turning dΘ of the computed output W given by dR/R which is a constant equal to 1/Γ where Γ is the golden mean. Further the successive values of the computed output W units of numerical length scale R follow the Fibonacci mathematical number series. Therefore, the value of k, the fractional departure from Euclidean geometrical shape of the computed output is derived from Eq. (3) as

$$k = \frac{1}{\Gamma^2} = 0.382 \qquad (4)$$

Since the steady state non-dimensional fractional departure from Euclidean shape of the strange attractor design traced in the phase space by the computed output W during successive length step increments is equal to 0.382, i.e. less than half, the overall Euclidean shape of the strange attractor is retained. Integrating Eq. (3) for computation starting with w_* units of length dR

$$W = \frac{w_*}{k}\ln Z \qquad (5)$$

The computed numerical output W follows a logarithmic spiral with Z equal to the scale ratio of the domain length R with respect to the yardstick length dR, i.e. Z = R/dR. The above concept of the growth of the computational structure W in successive length step increments dR is analogous to cellular automata computational technique and is identified as the universal period doubling route to chaos. The generation of strange attractors with selfsimilar fractal structures in computer realizations of nonlinear mathematical models is a direct consequence of computer precision related roundoff errors. The geometrical structure of the strange attractor is quantified by the recursion relation at Eq. (2). Eq. (2) is identified as the universal algorithm for the generation of the strange attractor pattern in computer realizations of nonlinear mathematical models.

2.3 Feigenbaum's constant for deterministic chaos

The Feigenbaum's constant a and d for the universal period doubling route to chaos may be derived directly from the universal recursion relation, i.e. Eq. (2) as shown in the following.

The Feignbaum's constant a is given by the successive spacing ratios W for adjoining period doublings. W and R are respective successive spacing ratio since by concept W and R are computed as incremental growth steps dW and dR for each stage of computation.

The recursion relation at Eq. (2) may be written as follows

$$2\left(\frac{WR}{w_* r}\right)^2 = \pi \left(\frac{W}{w_*}\right)^4 \left(\frac{R}{r}\right)^3 \qquad (6)$$

From Eq. (2) it can be shown that a = 1/k = relative increase in the computed output domain with respect to the yardstick length domain = Γ^2

a^2 =variance of a = Γ^4

$2a^2$ = Total variance of the relative fractional evolution of computed output domain for both clockwise and anticlockwise phase space trajectories.

The Feigenbaum's constant d is the successive spacing ratios of R for the universal recursion relation and can be shown to be equal to $W_2^4 R_2^3 / W_1^4 R_1^3$.

d is therefore equal to the relative volume intermittency of occurrence of Euclidean structure in the phase space during each computational step, i.e. $\pi/5$ radian angular rotation as shown earlier in Table 1. Therefore for one complete cycle of computation the relative volume inter mittency of occurrence of Euclidean structure in the computed phase space trajectory is πd. The reformulated universal recursion relation for numerical computation at Eq. (6) may now be written in terms of the universal Feigenbaum's constants as

$$2a^2 = \pi d$$

The above equation states that the relative volume intermittency of occurrence of Euclidean structure for one dominant cycle of computation contributes to the total variance of the fractional Euclidean structure of the strange attractor in the phase space of the computed numerical result.

3. CELL DYNAMICAL SYSTEM MODEL FOR ATMOSPHERIC FLOWS

In the following, theoretical consideration similar to those developed in Section 2 for deterministic chaos in numerical model results is advanced for coherent atmospheric flow structures (Mary Selvam 1988). In summary, the mean flow at the planetary atmospheric boundary layer (ABL) possesses an inherent upward momentum flux of frictional origin at the planetary surface. This turbulence scale upward momentum flux is progressively amplified by the exponential decrease of atmospheric density with height coupled with the buoyant energy supply by microscale fractional condensation on hygroscopic nuclei even in an unsaturated environment. The mean large scale upward momentum flux generates helical vortex roll (or large eddy) circulations in the planetary atmospheric boundary layer and is manifested as cloud rows/streets and mesoscale cloud clusters (MCC) in the global cloud cover pattern. A conceptual model of large and turbulent eddies is shown in Figure 2.

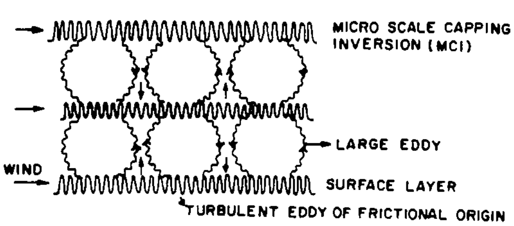

Figure 2 : Conceptual model of large and turbulent eddies in the ABL.

The generation of turbulent buoyant energy by the microscale fractional condensation is maximum at the crest of the large eddies and results in the warming of the large eddy volume. The turbulent eddies at the crest of the large eddies are identifiable by a microscale capping inversion (MCI) which rises upwards with the convective growth of the large eddy in the course of the day. This is seen as the rising inversion of the day time planetary boundary layer in the echosonde records. The space-time integrated mean of the turbulence scale vertical acceleration w_* generated by dominant eddy fluctuations of radius r give rise to large eddy acceleration W of radius R.

The above concept of large eddy growth from turbulences scale buoyant energy generation envisages large eddy growth in discrete length step increments dR equal to r and is therefore analogous to the 'cellular automata' computational technique where cell dynamical system growth occurs in unit length steps during unit intervals of time since turbulence scale yardstick for length and time are used for measuring large eddy growth. Large eddy growth by such length scale doubling is hereby identified as the universal period doubling route to chaos eddy growth process. Therefore for turbulent eddy acceleration w_* large eddy incremental growth is dR and is associated with large eddy acceleration dW and is given by Eq.1. The internal structure of large eddy circulations is made up of balanced small scale circulations tracing out the well known quasiperiodic Penrose tiling pattern identified as the quasicrystalline structure in condensed matter physics (see Table 1 and Figure 1).

As seen from Fig 1 and from the concept of eddy growth vigorous counter flow (mixing) characterises the large eddy volume. The steady state non-dimensional fractional volume dilution k of the large eddy volume by environmental mixing is given by Eq.3. Since the steady state non-dimensional fractional volume dilution of large eddy by inherent turbulent eddy fluctuations during successive length step increments is equal to 0.382, i.e., less than half, the overall Euclidean geometrical shape of the large eddy is retained as manifested in the cloud billows which resemble spheres.

The turbulent eddy circulation speed and radius increase with the progressive growth of large eddy as given by Eq.5 where the constant k equal to 0.382 is identified as the Von Karman's constant. The Von Karman's constant is therefore the universal constant for deterministic chaos.

4. CONCLUSION

The cell dynamical system model for deterministic chaos described in this paper enables to identify the quasiperiodic Penrose tiling pattern as the internal structure of the strange attractor design traced by digital computer realizations of nonlinear mathematical models of atmospheric flows as well as the observed real world atmospheric flows in the planetary atmospheric boundary layer. The Von Karman's constant equal to 0.38 is the universal constant for deterministic chaos which quantifies the steady state fractional departure from Euclidean geometry for computed and real world dynamical systems.

Acknowledgement

The author wishes to express her deep graditude to Dr.A.S.R.Murty for his keen interest and encouragement during the course of this study.

REFERENCES

Bak, P.C., C.Tang and K. Wiesenfeld, 1988 : Self-organized criticality, Phys. Rev.A, **38**, 364-374

Beck, C., and G.Roepstorff, 1987 : Effects of phase space discretization on the long-time behaviour of dynamical systems. Physica D., **25**, 173-180.

Janssen, T., 1988 : Aperiodic crystals : A contradictio in terminis ? Phys.Rep., **168**, 1-90.

Lovejoy, S., and D. Schertzer, 1986 : Scale invariance, symmetries, fractals and stochastic simulations of atmospheric phenomena. Bull.Amer.Meteorol.Soc., **67**, 21-32.

Mary Selvam, A., 1988 : The dynamics of deterministic chaos in numerical weather prediction models. Proc.8th Amer.Meteorol. Soc. Conf. Num.Wea.Pred., Baltimore, MD., USA. Feb.

Procaccia, I. 1988 : Universal properties of dynamically complex systems : The organization of chaos. Nature,**333**, 618-623.

Ruelle, D., and F.Takens, 1971 : On the nature of turbulence. Commun. Math. Phys.,**20**, 167-192; **23**, 343-344.

Townsend, A.A., 1956 : The structure of turbulent shear flow. Cambridge University Press., UK., 115-130.