Dynamical systems and nilpotent sub-Riemannian geometry

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We consider dynamical systems determined by distributions of real analytic vector fields, the equations of motion, and the associated gauge transformation are presented in detail. A gradation of the associated Lie algebras leads to the consideration of polynomial vector fields. The minimization of kinetic energy action in the class of horizontal curves associated with the distribution is formulated as a sub-Riemannian geodesic problem. Normal geodesics are fully described for the so-called Gaveau–Brockett distribution. The exponential mapping, the unitary spheres, and the wave fronts are calculated for particular cases. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2897031]

I. INTRODUCTION

Certain nonlinear dynamical systems can be described by means of families of vector fields that are usually called distributions. Such distributions encode the nonlinearities of the system and allow to write the corresponding differential equations. This situation is particularly transparent in the geometric theory of nonholonomic dynamical systems, see, for instance, Ref. 14. For distributions with a finite number of vector fields, the structure of the spanned Lie algebra, that is, the Lie algebra obtained by Lie bracketing iteratively the vector fields, determines most of the relevant properties of the system: existence of solutions, energy minimizing trajectories, etc. This is one of the reasons for which, for a given distribution, one expects to find another which is in certain sense equivalent, in such a way that the latter generates a nilpotent Lie algebra. The collection of techniques pointing out in that direction goes in the literature under the names of nilpotent approximations and nilpotentizations, see, for instance, the survey by Hermes[8] and the recent book by Montgomery.[13] The study of the variational problem of minimizing the kinetic energy for left invariant distributions on nilpotent Lie groups has been tackled in the literature under various approaches, from the theory of hypoelliptic operators in mathematical physics (see Refs. 5 and 6) to the so-called sub-Riemannian geometry, see (Ref. 13) passing of course through the formalism of the geometric theory of nonlinear control systems (see Ref. 2).

We study a nonlinear dynamical model defined in terms of distributions of polynomial vector fields satisfying the so-called bracket generating condition, such a condition guarantees the existence of trajectories provided the base manifold is connected. Examples of problems which can be expressed in these terms can be found in plasma physics[5] and nonholonomic mechanics.[13]

Let $\mathcal{M}$ be a smooth manifold and $T\mathcal{M}$ its tangent bundle, a distribution δ of rank n on $\mathcal{M}$ is a smooth rank n subbundle of $T\mathcal{M}$. Two flags of modules of vector fields are naturally associated with δ, namely, the *derived* and the *lower central* Lie flags, defined inductively as follows:

$$\delta^{(0)} = \delta, \quad \delta^{(i+1)} = \delta^{(i)} + [\delta^{(i)}, \delta^{(i)}], \quad \delta^{(0)} \subseteq \delta^{(1)} \subseteq \ldots \subseteq \delta^{(i)} \ldots ,$$

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respectively. In general, these flags are different. The distribution δ is said to be *bracket generating*, if for each $p \in \mathcal{M}$, there exists a positive integer m for which $\delta_p^{(m)} = T_p\mathcal{M}$. A point $p \in \mathcal{M}$ is said to be *regular* for δ, if all the $\delta^{(i)}$ have constant rank in a small enough neighborhood of p. An absolutely continuous curve $t \mapsto q(t)$ is said to be *horizontal*, if $\dot{q}(t) \in \delta(q(t))$, almost everywhere. For bracket generating distributions, Chow-Rashevski's theorem[13] guarantees that any $p, q \in \mathcal{M}$ can be connected by a horizontal curve, provided $\mathcal{M}$ is connected. A sub-Riemannian metric on $\mathcal{M}$ can be defined by considering a smooth varying inner product $q \mapsto \langle \cdot, \cdot \rangle_q$ in $\delta(q)$. For horizontal curves $t \mapsto \alpha(t)$, the length and the energy functionals are defined as customary,

$$\ell(\alpha) = \int \langle \dot{\alpha}, \dot{\alpha} \rangle^{1/2} \quad \text{and } \mathcal{E}(\alpha) = \frac{1}{2} \int \langle \dot{\alpha}, \dot{\alpha} \rangle, \tag{1}$$

respectively. If $\mathcal{M}$ is connected then the sub-Riemannian distance between $q_1, q_2 \in \mathcal{M}$ is well defined. $d(q_1, q_2) = \inf\{\ell(q) \mid q : [0, T_q] \to \mathcal{M}$ is horizontal and $q(0) = q_1, q(T_q) = q_2\}$, and the sub-Riemannian geodesic problem on $\mathcal{M}$ consists of the minimization of the functional ℓ in the class of horizontal curves. For curves parametrized by arc length, the variational problems for the functionals ℓ and $\mathcal{E}$ are equivalent.

Here, we shall be concerned only with distributions δ which are bracket generating and call $\mathfrak{g}$ the Lie algebra given by $\cup_i \delta^{(i)}$. This leads to a natural decomposition $\mathfrak{g} = H \oplus V$, in terms of the "horizontal" vector space $H = \delta$ and the "vertical" vector space V given by the complement. Remember for later use that for a Lie algebra $\mathfrak{g}$, we have the derived and the lower central series defined as follows:

$$\mathfrak{g}_{(0)} = \mathfrak{g}, \quad \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}], \quad \mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \cdots \supseteq \mathfrak{g}_{(i)} \cdots,$$

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_0, \mathfrak{g}_i], \quad \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_i \cdots.$$

A Lie algebra is called solvable if $\mathfrak{g}_{(i)} = 0$ for a certain integer i and it is called step j nilpotent if $\mathfrak{g}_j = 0$ for a certain integer $j > 1$. All nilpotent Lie algebras are solvable since $\mathfrak{g}_{(i)} \subseteq \mathfrak{g}_{2^i}$. In consequence, if $\mathfrak{g}$ is nilpotent we have a sequence of subalgebras according to its *solvable length*, given by the smallest integer i for which $\mathfrak{g}_{(i)} = 0$, which is at most equal to $k+1$ for step 2^k algebras.

In this work, we consider the nonlinear dynamical systems given by bracket generating distributions of real analytic vector fields. We approach the problem as a sub-Riemannian geodesic problem on the underlying manifold; we integrate the geodesic equations and describe some of the associated geometry.

Apart from this introduction, this paper contains six sections. In Sec. II, we set the dynamical problem that plays the role of archetype of the theory and is related to variational problems in nonlinear dynamics. For this case, we depict the Lie algebraic structure and exhibit a gauge transformation under which the variational equations remain invariant. In Sec. III, we propose a coordinate-free method for step-$(m+1)$ nilpotent Lie algebras $\mathfrak{g}$ with $\mathfrak{g}_{(2)} = 0$. We write explicitly a Philip Hall basis for the Lie algebra that clarifies by means of a gradation the hierarchy of dynamical systems proposed by Brockett and Dai in Ref. 3. We develop also some low dimensional examples. In Secs. IV–VII, we discuss the first degree of the gradation and obtain explicit expressions for the geodesics (Sec. V). The obtained formulas, allow us to present some of the sub-Riemannian geometry of the problem, in particular, the characterization of unit spheres, the complete parametrization of the exponential mapping (Sec. VI), and the conjugate locus in Sec. VII.

II. THE MODEL VARIATIONAL PROBLEM

In this section, we consider the nonlinear dynamical system in $\mathbb{R}^{n+1}$, determined by a bracket generating distribution Δ of real analytic vector fields $X_1, \ldots, X_n$, given as

$$X_i = \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial y}, \quad i = 1, \ldots, n, \tag{2}$$

where $\xi_i(x)$ are the real analytic functions in n variables. The horizontal curves, i.e., absolutely continuous curves $t \mapsto q(t)$ such that $\dot{q}(t) \in \text{spanspan}\{X_1(q(t)), \ldots, X_n(q(t))\}$, are solutions of the following differential system:

$$\dot{q} = \sum_{i=1}^{n} \dot{x}_i X_i(q), \tag{3}$$

where $q = (x, y)^T \in \mathbb{R}^{n+1}$, and y satisfies

$$\dot{y} = \sum_{i=1}^{n} \dot{x}_i \xi_i(x).$$

For the first degree Lie brackets, we write

$$X_{ij} := [X_i, X_j] = F_{ij} \frac{\partial}{\partial y}, i, \quad j = 1, \ldots, n,$$

with the antisymmetric matrix elements F_{ij}, defined as

$$F_{ij} = \frac{\partial}{\partial x_j} \xi_i - \frac{\partial}{\partial x_i} \xi_j.$$

It is easy to verify that the Jacobi identity reads as follows:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$

Observe that that since the ξ_j do not depend on y, all fields

$$\text{ad}_{X_{i_1}} \text{ad}_{X_{i_2}} \cdots \text{ad}_{X_{i_k}}(X_j) \quad \text{for } k > 0$$

commute among themselves. This is important since then the Lie algebra $\mathfrak{g}$ spanned by the distribution Δ, that is, $\mathfrak{g} = \cup_i \Delta^{(i)}$, is solvable with derived series $\mathfrak{g} = \mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} = \{0\}$.

Further, $\mathfrak{g}$ is filtered ($\Delta^{(i)} \subseteq \Delta^{(j)}$ for $i \leqslant j$, and $[\Delta^{(i)}, \Delta^{(j)}] \subseteq \Delta^{(i+j)}$) and graded ($\mathfrak{g} = \Sigma_{i \geqslant 0} \oplus \Delta_i$ with $\Delta_{i+1} = [\Delta, \Delta_i]$ and $\Delta^{(i)} = \Sigma_{j=0}^{i} \Delta_j$), for details on these definitions see, for instance, (Ref. 9). We shall call the parts of Δ_i to be *homogeneous of degree i*.

Together with the anticommutativity of the Lie bracket, the Jacobi identity bounds the number of linearly independent vector fields generated for the derived and the central flags. To see this, let us define the fields of *depth r* as follows:

$$X_\kappa = \text{ad}_{X_{\kappa_r}} \text{ad}_{X_{\kappa_{r-1}}} \cdots \text{ad}_{X_{\kappa_2}}(X_{\kappa_1}), \quad \kappa = (\kappa_r, \kappa_{r-1}, \ldots, \kappa_1), \quad r \geqslant 1,$$

with integers $\kappa_j = 1, \ldots, n$. If all fields are distinct and of depth one, the Jacobi identity is a (three terms) condition relating fields of depth three. For more than one field of depth two, the identity is trivial. Moreover, when two fields, say, X_i and X_j, are of depth one, and the third one, say, X, is of depth $s \geqslant 1$ then

$$[X_i, [X_j, X]] = [X_j, [X_i, X]],$$

expression that relates fields of depth $s+2$, which is just the statement of the commutativity of the partial differentiation, $\partial_i \partial_j = \partial_j \partial_i$. In conclusion, the Jacobi identity states that for $r > 3$ the order of the κ_i for $i > 3$ in X_κ is irrelevant.

Let now G be the simply connected Lie group associated with the Lie algebra $\mathfrak{g}$, in such a way that the X_i are *left invariant* vector fields. We can define a smooth varying inner product $\langle \cdot, \cdot \rangle$ at each plane $\Delta(q)=\text{span}\{X_1(q),\ldots,X_n(q)\}$ by declaring $\{X_i(q)\}$ to be an orthonormal set.

The distribution Δ determines a variational problem on G. energy-minimizing horizontal curves $t\mapsto q(t)$ satisfy $\langle \dot{q},\dot{q}\rangle=\langle \dot{x},\dot{x}\rangle$; for that reason, it shall be convenient to introduce the angles $0\leq\phi_i<2\pi$ and radii R_i, with $i=1,\ldots[n/2]$ such that

$$\dot{x}_{2i-1}=R_i\cos\phi_i,\quad \dot{x}_{2i}=R_i\sin\phi_i. \tag{4}$$

For n odd, an extra radius

$$\dot{x}_n=R_{[n/2]+1} \tag{5}$$

shall be necessary, in any case $\langle \dot{x},\dot{x}\rangle=\Sigma_i R_i^2=1$, and therefore the $\{R_i\}$ can be parametrized using spherical coordinates.

The standard variational method (see, for example, Ref. 4) consists in the study of the Lagrangian,

$$L=\frac{\lambda_0}{2}\|\dot{x}\|^2+\lambda(\dot{y}-\langle\xi,\dot{x}\rangle).$$

There are two classes of extremal curves, the ones for which $\lambda_0=0$, called abnormal or singular, and the ones for which $\lambda_0\neq 0$, usually called normal. For the latter, we can set without loss of generality that $\lambda_0=1$; the Euler–Lagrange equations in this case are the following:

$$\ddot{x}=\lambda F\dot{x},\quad \dot{\lambda}=0,$$

where F is the skew-symmetric matrix with entries F_{ij}. Clearly, the Lagrange parameter λ is a constant of motion. The Euler–Lagrange equations are invariant with respect to the gauge transformation

$$\xi_i\mapsto\xi_i+\frac{\partial\phi}{\partial x_i},$$

for a sufficiently smooth function $\phi(x)$. The Lagrangian and the fields are mapped to

$$L-\lambda\frac{d\phi}{dt}\quad\text{and } X_i+\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial y},$$

respectively, and the Lie algebra remains invariant. In terms of differential forms, the connection 1-form of Δ is

$$A=dy-\sum_j\xi_j dx_j.$$

Clearly $(A,X_i)=0$. The curvature 2-form of the distribution is

$$\mathcal{F}=2dA=\sum_{j,k}F_{jk}dx_j\wedge dx_k.$$

Here, $d\mathcal{F}=0$ is an analog of the homogeneous Maxwell equations (Jacobi identity) for this problem. Further, A is a higher dimensional analog of the *vector potential* and dA is the corresponding analog of the *magnetic field*. The above gauge transformation leads to

$$A\mapsto A-d\phi\quad\text{and } \mathcal{F}\mapsto\mathcal{F}.$$

Therefore, by means of gauge transformations, we can add to our convenience exact differentials leaving both the equations of motion and the curvature invariant.

For real analytic vector fields the resulting Lie algebra could be infinite in general, which cannot be our case, since we are assuming that the distribution Δ is bracket generating. Therefore, the vector fields must generate finite Lie algebras.

III. GRADED NILPOTENT LIE ALGEBRAS

The distribution introduced above can be studied by considering the Taylor series expansion around the origin

$$\xi_j(x) = \sum_{i_1,\ldots,i_n\geqslant 0} \frac{1}{i_1!i_2!\cdots i_n!}\frac{\partial^{i_1}}{\partial x_1^{i_1}}\cdots\frac{\partial^{i_n}}{\partial x_n^{i_n}}\xi_j(x)\bigg|_{x=0} x_1^{i_1}\cdots x_n^{i_n}.$$

The general analytic case has been studied among others by Lafferriere and Sussmann.[11] Brockett and Dai.[3] have proposed a hierarchy of nonlinear dynamical systems by considering a finite number of terms in the Taylor series, that is, they take polynomial vector fields in particular coordinate systems. Clearly if the ξ_i are polynomials of degree at most m, then the resulting Lie algebra $\mathfrak{g}$ shall be finite as required. Furthermore, $\mathfrak{g}$ shall be nilpotent of order of nilpotency m (i.e., step $m+1$), since then $\Delta_k=0$ for all $k>m$. We propose here a *coordinate-free* method to study this problem.

We consider a general approach that leads naturally to the classification problem of isomorphism classes of finite dimensional nilpotent Lie algebras. Furthermore, our results provide a coordinate-free method for studying nonlinear dynamical systems. We consider a rank n distribution Δ, as that given in the preceding section, but for general linearly independent polynomial functions ξ_i of degree m. Denote the Lie algebra generated by Δ as $\mathfrak{g}$. This Lie algebra is filtered, graded (recall $\Delta^{(i)}=\Sigma_{j=0}^{i}\Delta_j$), and nilpotent of step-$(m+1)$. Further, $\mathfrak{g}$ has solvable length two, i.e., $\mathfrak{g}_{(2)}=0$. The skew symmetry of the Lie bracket together with the Jacobi identity implies that not all commutators can be linearly independent. A generic formulation of the problem for nilpotent Lie algebras would be reached once a basis for the commutators is explicitly given. For then the underlying simply connected Lie group can be obtained by means of the exponential map and the BCH formula, privileged coordinates can be used to write a canonical basis of invariant vector fields.

Lemma: The number of linearly independent vector fields in Δ_r *is* $D_{n,r}=r\binom{n+r-1}{r+1}$ *for* $r>0$. *Furthermore, the Lie algebra* $\mathfrak{g}$ *has dimension* $n+1+D_{n+1,m}-\binom{n+m}{m}$.

Proof: It is clear that after m brackets the resulting vector fields are central. Since the fields of distinct depths are linearly independent, the dimensions of the subspaces Δ_r can be computed directly by consideration of the Jacobi identity as follows.

Assume that there are $D_{n,m-2}$ fields X of depth $m-1$ for $m\geqslant 2$. They produce then $nD_{n,m-2}$ fields $[X_i,X]$ of depth m. However, there are $n(n-1)/2$ Jacobi identities

$$[X_j,[X_i,X']] = [X_i,[X_j,X']],$$

for each of the $D_{n,m-3}$ fields X' of depth $m-2$, relating fields of depth m, which must be subtracted. However, again, we must consider the $n(n-1)(n-2)/3!$ Jacobi identities,

$$[X_j,[X_i,[X_k,X'']]] = [X_i,[X_j,[X_k,X'']]] = [X_i,[X_k,[X_j,X'']]],$$

which are anyway satisfied for each of the $D_{n,m-4}$ fields X'' of depth $m-3$. These identities must be subtracted from those relating the fields of depth $m-2$, and hence added to the total sum. For then, following the same reasoning, the general recurrence formula is

$$\sum_{k=0}^{m} (-)^k \binom{n}{k} D_{n,m-k-1} = 0, \quad m\geqslant 2.$$

The initial conditions are as follows. First, $D_{n,0}=0$, since there are no two terms Jacobi identities relating fields of depth m and containing only m fields of Δ. Second, $D_{n,-1}=-1$, because of the identity with m fields in Δ and finally $D_{n,-s}=0$, $s>1$, because there are no deeper relations. The solution of the formula is, in fact, $D_{n,m}$, as given above. This can be corroborated by deducing first the generating function $(n-1/x)/(1-x)^n$ for the $D_{n,m}$. The dimension of the algebra can be obtained using the identity

$$\sum_{k=0}^{m} \binom{n+k-1}{k} = \binom{n+m}{m}.$$

□

Note the difference with Witt's dimension formula for *free* Lie algebras[9] for $r\geqslant 93$. Once we know the dimension of $\mathfrak{g}$ we shall describe precisely a Philip Hall basis.[1,11,14] Let us recall that such a basis consists of a totally ordered set $\{\mathcal{P},<\}$ given by the following:

(1) The X_i belong to $\mathcal{P}$.
(2) If $A,B\in\mathcal{P}$ and length $(A)<\text{length}(B)$, then $A<B$.
(3) If C is not in Δ, then $C\in\mathcal{P}$ iff $C=[A,B]$ with $A,B\in\mathcal{P}$, $A<B$ and either $B\in\Delta$ or $B=[D,E]$, with $D,E\in\mathcal{P}$, $D\leqslant A$ and $D<E$.

Evidently in our case the total order $<$ is determined by the depth of the bracket.

Proposition: A Philip Hall basis $\mathcal{P}$ of the step-$(m+1)$ nilpotent Lie algebra $\mathfrak{g}$ is given as follows.

(1) *The n elements of Δ and the $D_{n,1}=n(n-1)/2$ fields of depth two $X_{i_1,i_2}=[X_{i_1},X_{i_2}]$, for $i_1<i_2$.*
(2) *$X_{i_3i_1i_2}=ad_{X_{i_3}}ad_{X_{i_1}}X_{i_2}$, for $i_1<i_2\leqslant i_3$ and $X_{i_2i_1i_3}=ad_{X_{i_2}}ad_{X_{i_1}}X_{i_3}$, for $i_1\leqslant i_2<i_3$. These are a total of $D_{n,2}=n(n^2-1)/3$ linearly independent fields of depth three.*
(3) *The $D_{n,r+2}$ fields of depth $r+3>3$ for $r=1,\ldots,m-3$, $X_{ji}=ad_{X_{j_r}}\cdots ad_{X_{j_1}}ad_{X_{i_3}}ad_{X_{i_1}}X_{i_2}$, and $X_{ji'}=ad_{X_{j_r}}\cdots ad_{X_{j_1}}ad_{X_{i_2}}ad_{X_{i_1}}X_{i_3}$, with $j_1\leqslant j_2\leqslant\cdots\leqslant j_r$.*

Proof: For (1), step two or larger, there is no restriction from the Jacobi identity and clearly $\mathcal{P}$ is given by Δ and X_{i_1,i_2} with $i_1<i_2$. There are $n(n-1)/2=D_{n,1}$ elements X_{ij} as expected. For (2), step three or larger, notice first that for $i_3>i_2>i_1$, the Jacobi identity for all three fields in Δ reads

$$[X_{i_3},[X_{i_2},X_{i_1}]]+[X_{i_2},[X_{i_1},X_{i_3}]]+[X_{i_1},[X_{i_3},X_{i_2}]]=0.$$

Here, the first two terms (X_{i_3,i_1,i_2} and X_{i_2,i_1,i_3}) belong to $\mathcal{P}$, but not the third X_{i_1,i_3,i_2}. Therefore, at this depth the conditions for $\mathcal{P}$ take into account the Jacobi identities. Thus there are $2n(n-1)\times(n-2)/3!$ elements $X_{i_3i_1i_2},X_{i_2i_1i_3}$ for $i_1<i_2<i_3$ and $2n(n-1)/2$ elements $X_{i_1i_1i_2},X_{i_2i_1i_2}$ with $i_1<i_2$, giving a total of $n(n^2-1)/3=D_{n,2}$ basis elements of depth three as expected. From step four upward the Jacobi identity is nontrivial only if exactly two fields belong to Δ, in that case $[X_{i_2},[X_{i_1},X]]+[X_{i_1},[X,X_{i_2}]]=0$, for $i_1<i_2$ and $X_j<X$ for all $X_j\in\Delta$. Then, the first term belongs to $\mathcal{P}$ but not the second. Again, the condition for the elements of $\mathcal{P}$ take fully into account the Jacobi identity for *all* higher depths. For depth $r+3>3$, there are two subclasses:

(1) For X_{ji} there is a single set of fields with $r+3$ distinct subindices: $j_r>\cdots>j_1>i_3>i_2>i_1$ and for $X_{ji'}$ there are $r+1$ sets of linearly independent fields with $r+3$ distinct subindices: $j_r>\cdots>j_2>j_1>i_3>i_2>i_1;j_r>\cdots>j_2>i_3>j_1>i_2>i_1;\cdots;i_3>j_r>\cdots>j_1>i_2>i_1$. In this subclass there are a total of $(r+2)\binom{n}{r+3}$ linearly independent fields of depth $r+3$.
(2) In this subclass two or more subindices are equal; there is at least one equality and at most $r+1$. For X_{ji}, i_1 is always distinct and $j_r\geqslant\cdots\geqslant j_1\geqslant i_3\geqslant i_2>i_1$. For $X_{ji'}$ there are $r+1$ types of inequalities: $j_r\geqslant\cdots\geqslant j_1\geqslant i_3>i_2\geqslant i_1$, $j_r\geqslant\cdots\geqslant j_2\geqslant i_3>j_1\geqslant i_2\geqslant i_1,\cdots,i_3>j_r\geqslant\cdots\geqslant j_1\geqslant i_2\geqslant i_1$. Now, s equalities lead to a relation with only $r+2-s$ inequalities and each corresponds to $\binom{n}{r+3-s}$ distinct indices. However, the $s=1,\ldots,r+1$ equalities can occur in $\binom{r+1}{s}$ ways times the $r+2$ types of inequalities.

Both subclasses give together the number of linearly independent fields of depth $r+3>3$ as

$$(r+2)\sum_{s=0}^{r+1}\binom{r+1}{s}\binom{n}{r+3-s}=(r+2)\binom{n+r+1}{r+3},$$

but this is precisely $D_{n,r+2}$. □

The remaining elements of $\mathfrak{g}$ are obtained using the skew symmetry and the Jacobi identity. The question about the possible subalgebras of a fixed step and their classification shall not be taken into account, since it is not relevant for a general setting. The exponential map and the BCH formula allow now in principle to obtain in terms of the above Philip Hall basis the Lie group law and from it the associated left invariant vector fields in canonical coordinates. These coordinates would correspond to the so called "Philip Hall coordinates,"[11] defined for analytical vector fields.

Example 1: For the step-2 algebra in n variables, up to isomorphisms, the basis is

$$[X_i,X_j]=X_{ij}, \quad i<j \quad i,j=1,\ldots,n. \tag{6}$$

The remaining nontrivial elements of the algebra are $X_{i_2,i_1}=-X_{i_1,i_2}$.

Example 2: For step-3 the basis is given by

$$[X_{i_1},X_{i_2}]=X_{i_1 i_2}, \quad i_1<i_2,$$

$$[X_{i_3},X_{i_1 i_2}]=X_{i_3,i_1 i_2}, \quad i_1<i_2\leqslant i_3, \quad [X_{i_2},X_{i_1 i_3}]=X_{i_2,i_1 i_3}, \quad i_1\leqslant i_2<i_3,$$

The remaining elements of the algebra are again $X_{i_2,i_1}=-X_{i_1,i_2}$ *and*

$$[X_{i_1},X_{i_2 i_3}]=X_{i_3,i_1 i_2}-X_{i_2,i_1 i_3}, \quad i_1<i_2<i_3,$$

with $i_1,i_2,i_3=1,\ldots,n$.

Example 3: For only two variables but step-$(m+1)$*, a Philip Hall basis is*

$$[X_1,X_2]=X_{12}, \quad \mathrm{ad}_{X_{j_k}}\mathrm{ad}_{X_{j_{k-1}}}\cdots\mathrm{ad}_{X_{j_1}}X_{12}=X_{j,12},$$

The remaining elements of the algebra are either zero or are

$$\mathrm{ad}_{X_{\pi(j_k)}}\mathrm{ad}_{X_{\pi(j_{k-1})}}\cdots\mathrm{ad}_{X_{\pi(j_1)}}X_{12}=X_{j,12},$$

with $j=(j_k,\ldots,j_1)$ *for* $j_1\leqslant j_2\leqslant\ldots\leqslant j_k$, $j_k=1,2$, $k=2,\ldots,m$ *and* $\pi(j)$ *is any permutation of the elements of j and all other antisymmetric expressions.*

The associated sub-Riemannian nilpotent problem is therefore given in terms of an inner product and the distribution Δ, which generates a step-$(m+1)$ nilpotent Lie algebra $\mathfrak{g}$ of dimension as given in the last lemma. There is a natural principal bundle $(P,\mathcal{M},G_0)$, with total space P of the same dimension as $\mathfrak{g}$, base space $\mathcal{M}$ of dimension n, and Abelian structure group G_0, given by the Lie group of the subalgebra generated by all basis elements of $\mathfrak{g}$ not in Δ. The differential system for $q\in P$ defines the horizontal lifts of tangent vectors on $T\mathcal{M}$. In the next sections, the step-2 problem is analyzed in detail. Some additional results concerning the step-$(m+1)$ case will be reported elsewhere including explicit coordinated bases.

IV. THE FIRST DEGREE OF THE GRADATION

As explained above, the first degree of the gradation corresponds to a distribution $\Delta=\{X_i,\ldots,X_n\}$ for which the only nonzero brackets are $[X_i,X_j]=X_{ij}$ with $i<j$, $i,j=1,\ldots,n$. According to the last lemma, the Lie algebra $\mathfrak{g}$ generated by Δ is filtered, graded, nilpotent of step 2, and has dimension $n+\frac{1}{2}n(n-1)$. We can choose coordinates $\{q=(x,z)=(x_1,\ldots,x_n,z_{12},\ldots,z_{(n-1)n})\in\mathbb{R}^n\times\mathfrak{so}_n\}$ for which $X_i=\partial/\partial x_i+\Sigma_{j\neq i}x_j\partial/\partial z_{ij}$, together with $X_{ij}=-2\partial/\partial z_{ij}$ realize a Philip Hall basis for $\mathfrak{g}$. Furthermore, for any horizontal curve $t\mapsto q(t)$, $(\dot{q}(t)\in\Delta(q(t))\text{a.e.})$, $\dot{q}=(\dot{x},x\wedge\dot{x})$.

A sub-Riemannian metric is obtained by declaring $\{X_i(q)\}$ to be an orthonormal set at each $q=(x,z)\in\mathbb{R}\times\mathfrak{so}_n$. Energy minimizing trajectories turn out to be extremals corresponding to the Lagrangian,

$$L=\frac{\lambda_0}{2}\|\dot{x}\|^2-\frac{1}{2}\mathrm{Tr}\,\Lambda(\dot{z}-x\wedge\dot{x}),\tag{7}$$

where $\Lambda=(\lambda_{ij})$ is the skew-symmetric matrix of Lagrange multipliers.

In these coordinates, the associated extremal problem is known as the *Gaveau Brockett problem*. Two situations can occur. Either $\lambda_0=0$, which leads to the *singular* or *abnormal case*.[12] The second situation corresponds to $\lambda_0\neq 0$, the so-called *normal case*. In this last case, we can set $\lambda_0=1$, without loss of generality. In the Gaveau–Brockett problem, there are not strictly abnormal extremals. It can be shown that any abnormal extremal can also be obtained as projection of a normal one. We shall consider here only normal extremals.

In this case, the Euler–Lagrange equations $\ddot{x}=\Lambda\dot{x}$ and $\dot{\Lambda}=0$ lead us to the conclusion that the λ_{ij} are constants of motion and that

$$\frac{d}{dt}(\dot{x}-\Lambda x)=0.$$

However for $x(0)=0$, we obtain therefore that the initial velocity components $\dot{x}-\Lambda x=\dot{x}_0$ constitute a set of n constants of motion. These correspond essentially to the right invariant vector fields.

In the abnormal case, the Euler–Lagrange equations are $\Lambda\dot{x}=0$; therefore for Λ nonsingular $x=$const, whereas for the case with a single zero eigenvalue $\dot{x}=(0,\ldots,0,\pm 1)$, according to the normalization $\langle\dot{x},\dot{x}\rangle=1$.

V. NORMAL EXTREMALS

From here on, assume that $(x,z)\in R^n\times\mathfrak{so}_n$ is a geodesic arc defined in certain interval, with initial condition $(x(0),z(0))=(0,\mathbf{0})$, and also that (x,z) is the projection of a normal extremal $(x,z,\dot{x},\Lambda)$ with Λ satisfying that for n even Λ has all its eigenvalues different from zero, and for n odd it has only one zero eigenvalue. For simplicity, we shall assume that the eigenvalues are nondegenerated.

Let $\sigma\subset i\mathbb{R}$ be the spectrum of Λ, and let π be its characteristic polynomial. Each $\mu\in\sigma$ determines a $n\times n$ complex matrix $\pi_\mu=\Pi_{\nu\in(\sigma-\{\mu\})}(\Lambda-\mathbf{I}\nu)/\pi'(\mu)$, which is the *spectral projector* associated with μ. They are Hermitian matrices, and for n odd, the projector corresponding to the eigenvalue zero is a *real*, symmetric $n\times n$ matrix denoted as π_0.

The Lagrange–Sylvester formula for matrix functions readily yields the spectral formula

$$x=\sum_{\mu\in\sigma}\frac{1}{\mu}(e^{\mu t}-1)\pi_\mu\dot{x}_0,\tag{8}$$

for n even plus the additional term $t\pi_0\dot{x}_0$ for n odd. Write now the nonzero eigenvalues of Λ as the elements of the ordered set

$$\{i\lambda_1,-i\lambda_1,\ldots,i\lambda_{\lfloor n/2\rfloor},-i\lambda_{\lfloor n/2\rfloor}\}\quad\text{with }\lambda_k>0\quad\text{for all }k,\tag{9}$$

correspondingly; the eigenvectors are the elements of the ordered set $\{v_1,v_{-1},\ldots,v_{\lfloor n/2\rfloor},v_{-\lfloor n/2\rfloor}\}$ and the projectors the elements of the ordered set $\{\pi_1,\pi_{-1},\ldots,\pi_{\lfloor n/2\rfloor},\pi_{-\lfloor n/2\rfloor}\}$. Observe that $v_{-k}=\bar{v}_k$ for all k. Denote by π_0 the projector for the zero eigenvalue and by v_0 its (real) eigenvector v_0, with $v_0\in\ker\Lambda$. From the definition, one obtains

$$\pi_i v_j=\delta_{ij}v_i\quad\text{for all } i \text{ and } j.\tag{10}$$

Assume now that n is even, the Cayley–Hamilton theorem implies

$$\pi_i \pi_j = \delta_{ij} \pi_i \quad \text{for all } i \text{ and } j, \tag{11}$$

but then the set $\{v_1, v_{-1}, \ldots, v_{n/2}, v_{-n/2}\} \subset \mathbb{C}^n$ is orthonormal with respect to the standard Hermitian product $\langle v, w\rangle = v^\dagger w$. In fact, if $i \neq j$ then

$$\langle v_i, v_j\rangle = \langle \pi_i v_i, \pi_j v_j\rangle = \langle v_i, \pi_i \pi_j v_j\rangle = \delta_{ij}. \tag{12}$$

Decompose now

$$v_j = \frac{1}{\sqrt{2}}(\alpha_{2j-1} + i\alpha_{2j}),$$

where the α_k are real unit orthogonal vectors, i.e., $\alpha_i \cdot \alpha_j = \delta_{ij}$, where $\cdot$ denotes the standard Euclidean product in $\mathbb{R}^n$. Since $v_k \in \ker(\Lambda - i\lambda_k \mathbf{I})$ for each $k = 1, \ldots, \lfloor n/2 \rfloor$, then

$$\alpha_{2k-1} = \frac{1}{\lambda_k}\Lambda \alpha_{2k} \quad \text{and } \alpha_{2k} = -\frac{1}{\lambda_k}\Lambda \alpha_{2k-1} \quad \text{for } k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{13}$$

Note further that α_k are eigenvectors of Λ^2 with eigenvalue $-\lambda_k^2$. In conclusion, $\{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal basis of $\mathbb{R}^n$. A similar analysis can be done for n odd, by considering the linearly independent orthogonal set $\{v_1, v_{-1}, \ldots, v_{\lfloor n/2\rfloor}, v_{-\lfloor n/2\rfloor}\}$, together with the (real) eigenvector v_0, to obtain the orthogonal basis $\{\alpha_1, \ldots, \alpha_{n-1}, v_0\}$, for the space of $\mathbb{R}^n$. Since $\dot{x}_0 \in \mathbb{R}^n$ is a (real) fixed vector, it can be verified that

$$\text{span}\{\text{Re}(\langle \dot{x}_0, v_k\rangle v_k), \text{Im}(\langle \dot{x}_0, v_k\rangle v_k)\} = \text{span}\{\text{Re}(v_k), \text{Im}(v_k)\},$$

for $k = 1, 2, \cdots, \lfloor n/2 \rfloor$. As expected, the geodesic arcs can be easily given in terms of the wedge products of the basis $\{\alpha_k\}$, for n even, and additionally $\alpha_0 = v_0$, for n odd. Recall now the parametrization given by Eqs. (4) and (5) and make the associations

$$\langle \dot{x}_0, \alpha_{2k-1}\rangle = R_k \cos \phi_k, \quad \langle \dot{x}_0, \alpha_{2k}\rangle = R_k \sin \phi_k,$$

for n even, and additionally $\langle \dot{x}_0, v_0\rangle = R_{\lfloor n/2\rfloor+1}$, for n odd. These relations mean simply that the R_i are the lengths of the projections of $\dot{x}_0$ on span$\{\alpha_{2i-1}, \alpha_{2i}\}$ and for n odd, $R_{\lfloor n/2\rfloor+1}$ is the projection in the direction of v_0.

Theorem: *Let $(x, z) \in \mathbb{R}^n \times \mathfrak{so}_n$ be a geodesic arc, for n even,*

$$x = \sum_{i=1}^{\lfloor n/2\rfloor} \bar{x}_{2i-1}\alpha_{2i-1} + \bar{x}_{2i}\alpha_{2i}, \tag{14}$$

$$z = \sum_{i,j=1}^{\lfloor n/2\rfloor} \bar{z}_{2i,2j}\alpha_{2i} \wedge \alpha_{2j} + \bar{z}_{2i,2j-1}\alpha_{2i} \wedge \alpha_{2j-1} + \bar{z}_{2i-1,2j-1}\alpha_{2i-1} \wedge \alpha_{2j-1}, \tag{15}$$

where

$$\bar{x}_{2i-1} = \frac{R_i}{\lambda_i}(\sin(\lambda_i t + \phi_i) - \sin \phi_i), \quad \bar{x}_{2i} = \frac{R_i}{\lambda_i}(\cos(\lambda_i t + \phi_i) - \cos \phi_i),$$

$$\bar{z}_{2i,2j} = \frac{R_i R_j}{\lambda_i}\left(\frac{\cos(\phi_i - \phi_j) - \cos((\lambda_i - \lambda_j)t + \phi_i - \phi_j)}{2(\lambda_i - \lambda_j)} + \frac{\cos((\lambda_i + \lambda_j)t + \phi_i + \phi_j) - \cos(\phi_i + \phi_j)}{2(\lambda_i + \lambda_j)} - \frac{\cos \phi_i(\cos(\lambda_j t + \phi_j) - \cos \phi_j)}{\lambda_j}\right),$$

$$\bar{z}_{2i,2j-1} = \frac{R_iR_j}{\lambda_i\lambda_j}\left(\frac{(\lambda_i+\lambda_j)[\sin((\lambda_i-\lambda_j)t+\phi_i-\phi_j)-\sin(\phi_i-\phi_j)]}{2(\lambda_i-\lambda_j)} + \frac{(\lambda_j-\lambda_i)[\sin((\lambda_i+\lambda_j)t+\phi_i+\phi_j)-\sin(\phi_i+\phi_j)]}{2(\lambda_i+\lambda_j)} + \sin\phi_j\cos(\lambda_i t+\phi_i) - \cos\phi_i\sin(\lambda_j t+\phi_j)\right),$$

$$\bar{z}_{2i,2i-1} = \frac{tR_i^2}{\lambda_i} - \frac{R_i^2\sin\lambda_i t}{\lambda_i^2},$$

$$\bar{z}_{2i-1,2j-1} = \frac{R_iR_j}{\lambda_i}\left(\frac{\cos(\phi_i-\phi_j)-\cos((\lambda_i-\lambda_j)t+\phi_i-\phi_j)}{2(\lambda_i-\lambda_j)} - \frac{\cos((\lambda_i+\lambda_j)t+\phi_i+\phi_j)-\cos(\phi_i+\phi_j)}{2(\lambda_i+\lambda_j)} - \frac{\sin\phi_i(\sin(\lambda_j t+\phi_j)-\sin\phi_j)}{\lambda_j}\right).$$

For n odd, we obtain the same summations, plus the extra terms $tR_{\lfloor n/2\rfloor+1}\alpha_0$ *for (14) and*

$$\sum_{i=1}^{\lfloor n/2\rfloor}\frac{R_iR_{\lfloor n/2\rfloor+1}}{\lambda_i}\left[-t(\sin(\lambda_i t+\phi_i)+\sin\phi_i)-\frac{2}{\lambda_i}(\cos(\lambda_i t+\phi_i)-\cos\phi_i)\right]\alpha_{2i-1}\wedge\alpha_0 + \frac{R_iR_{\lfloor n/2\rfloor+1}}{\lambda_i}\left[-t(\cos(\lambda_i t+\phi_i)+\cos\phi_i)+\frac{2}{\lambda_i}(\sin(\lambda_i t+\phi_i)-\sin\phi_i)\right]\alpha_{2i}\wedge\alpha_0,$$

for Eq. (15).

Proof: Consider first n even, thus for each $k\in[\![1,n/2]\!]$ we have

$$\pi_k\dot{x}_0 = \langle\dot{x}_0,v_k\rangle v_k \quad\text{and}\quad \pi_{-k}\dot{x}_0 = \langle\dot{x}_0,v_{-k}\rangle v_{-k}.$$

Therefore Eq. (8) yields

$$x = \sum_{k=1}^{\lfloor n/2\rfloor}\frac{1}{i\lambda_k}(e^{i\lambda_k t}-1)\langle\dot{x}_0,v_k\rangle v_k - \frac{1}{i\lambda_k}(e^{-i\lambda_k t}-1)\langle\dot{x}_0,v_{-k}\rangle v_{-k} = \sum_{k=1}^{\lfloor n/2\rfloor}\frac{1}{i\lambda_k}[(\cos(\lambda_k t)-1)(\langle\dot{x}_0,v_k\rangle v_k - \overline{\langle\dot{x}_0,v_k\rangle v_k})] + \frac{1}{i\lambda_k}[i\sin(\lambda_k t)(\langle\dot{x}_0,v_k\rangle v_k + \overline{\langle\dot{x}_0,v_k\rangle v_k})],$$

and

$$2\,\mathrm{Re}(\langle\dot{x}_0,v_k\rangle v_k) = R_j(\alpha_{2j-1}\cos\phi_j - \alpha_{2j}\sin\phi_j), \quad 2\,\mathrm{Im}(\langle\dot{x}_0,v_k\rangle v_k) = R_j(\alpha_{2j-1}\sin\phi_j + \alpha_{2j}\cos\phi_j), \tag{16}$$

which lead to Eq. (14). Thus

$$\dot{x} = \sum_{j=1}^{\lfloor n/2\rfloor}(-R_j)\sin(\lambda_j t+\phi_j)\alpha_{2j} + R_j\cos(\lambda_j t+\phi_j)\alpha_{2j-1},$$

and therefore an elementary integration of $\dot{z}=x\wedge\dot{x}$ yields the result, using Eq. (16). Assume now that n is odd, then

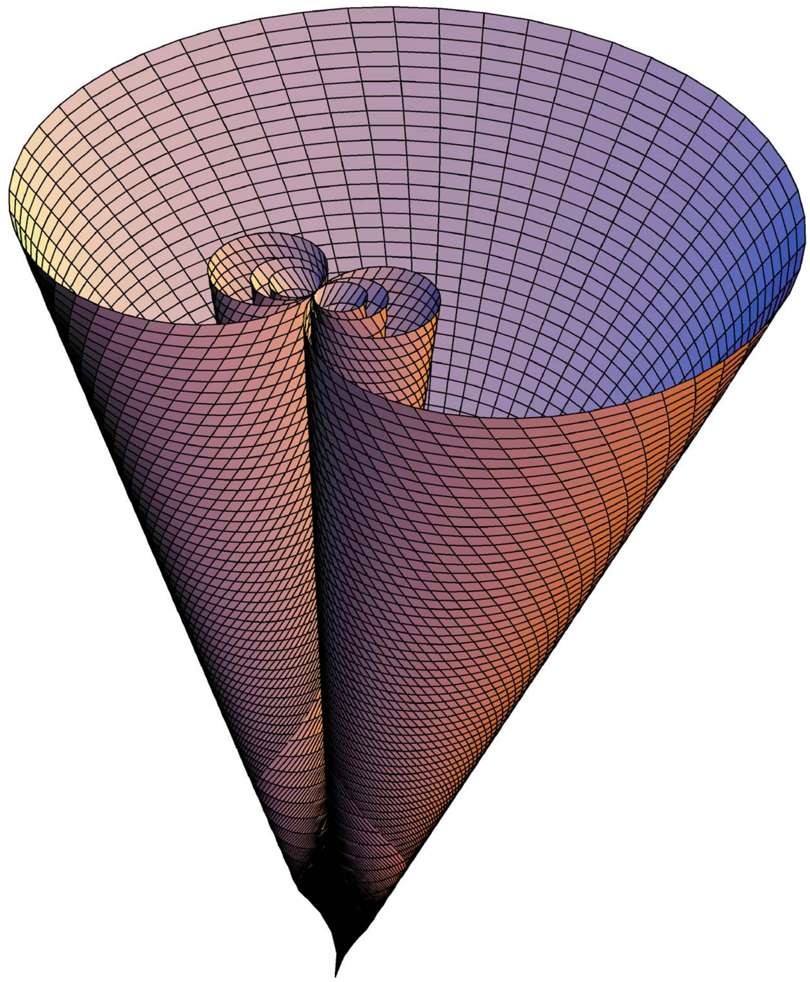

FIG. 1. (Color online) A set of trajectories in the subspaces generated by α_{2i-1}, α_{2i}, and α_0, as functions of time and λ_i.

$$\dot{x} = \gamma_0 + \sum_{j=1}^{\lfloor n/2 \rfloor} (-R_j)\sin(\lambda_j t + \phi_j)\alpha_{2j} + R_j \cos(\lambda_j t + \phi_j)\alpha_{2j-1},$$

with $\gamma_0 = \langle \dot{x}_0, \alpha_0 \rangle \alpha_0$. In consequence, once more, an elementary integration finishes the proof with Eq. (16). □

For arbitrary dimensions, the projections of the component x to the planes span$\{\alpha_{2i-1}, \alpha_{2i}\}$ are circles passing through the origin, and with radii R_i/λ_k. In odd dimensions, the projections of the component x to the three dimensional subspaces span$\{\alpha_{2i-1}, \alpha_{2i}, \alpha_0\}$ are helices, since x varies linearly in the direction of the vector α_0. In this case, one can write explicitly $t = \langle \dot{x}_0, \pi_0 x \rangle / \|\pi_0 \dot{x}_0\|^2$.

In Fig. 1, a set of trajectories is shown in the subspaces generated by α_{2i-1}, α_{2i}, and α_0. Each helix corresponds to a fixed λ_i, has radius R_i/λ_i, and hits the vertical α_0-axis at times $t = k_i 2\pi/\lambda_i$, for $k_i = 1, 2, \ldots$. In Fig. 2, the trajectories are shown as functions of time and ϕ_i in span$\{\alpha_{2i-1}, \alpha_{2i}, \alpha_{2i-1,2i}\}$ for fixed λ_i. Finally, in Fig. 3, a set of trajectories for several fixed values of λ_i is shown inside the unit sub-Riemannian sphere (see next section).

In this problem a geodesic arc which is projection of an abnormal extremal (not strictly abnormal) corresponds to a straight line. We can conclude that for n odd, the helix described above, which is projection of a normal extremal, has directrix parallel to the eigenspace corresponding to the eigenvalue zero, which coincides with the line given by the abnormal.

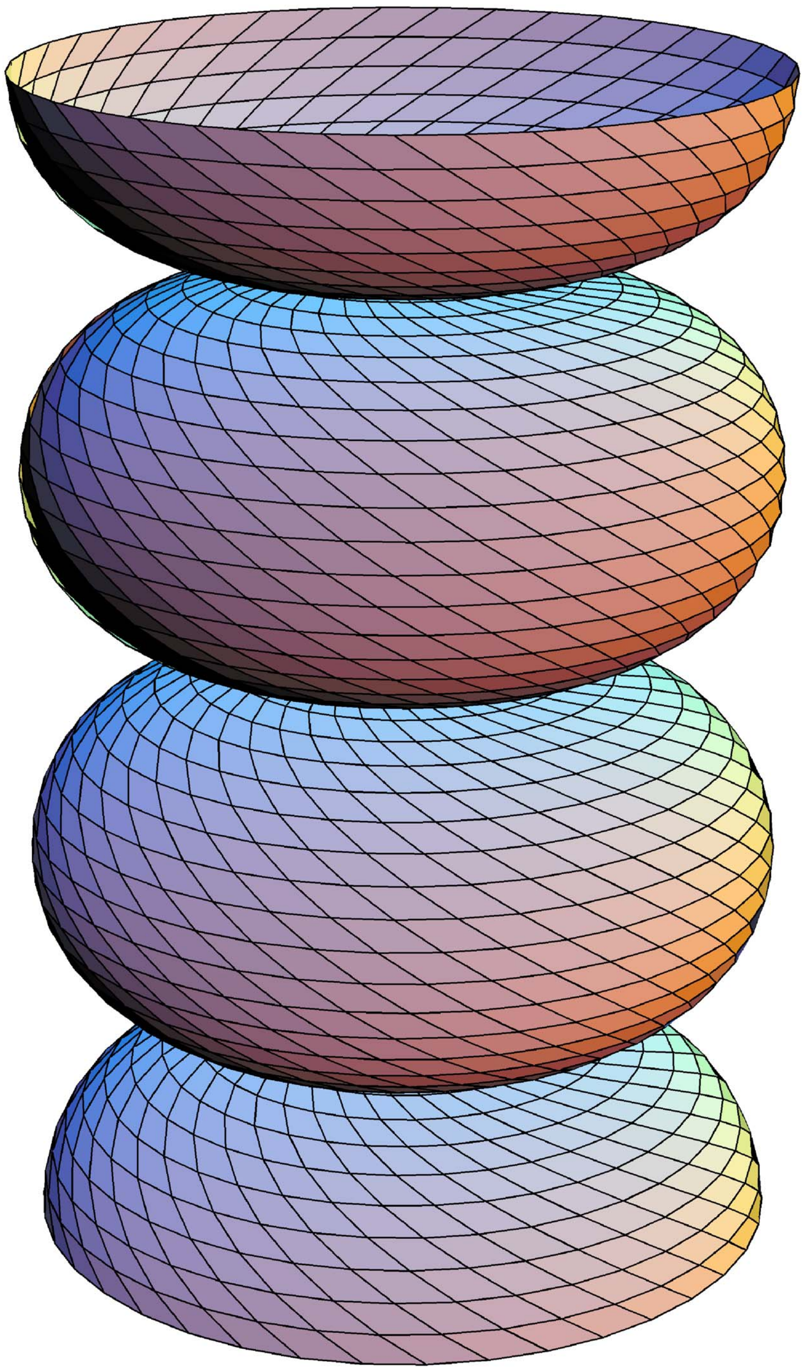

FIG. 2. (Color online) A set of trajectories in the subspaces generated by α_{2i-1}, α_{2i}, and $\alpha_{2i-1,2i}$, as functions of time and ϕ_i, for fixed λ_i.

The component x of the geodesic arc (x,z) can be given in a completely coordinate free manner, as the locus of certain algebraic surfaces, as follows.

Proposition: If (x,z) *is a geodesic arc then*

$$\Lambda^k x = \Lambda^{k-1}\dot{x} - \Lambda^{k-1}\dot{x}_0, \tag{17}$$

for $k=1,\ldots,\lfloor n/2 \rfloor$. *Furthermore,*

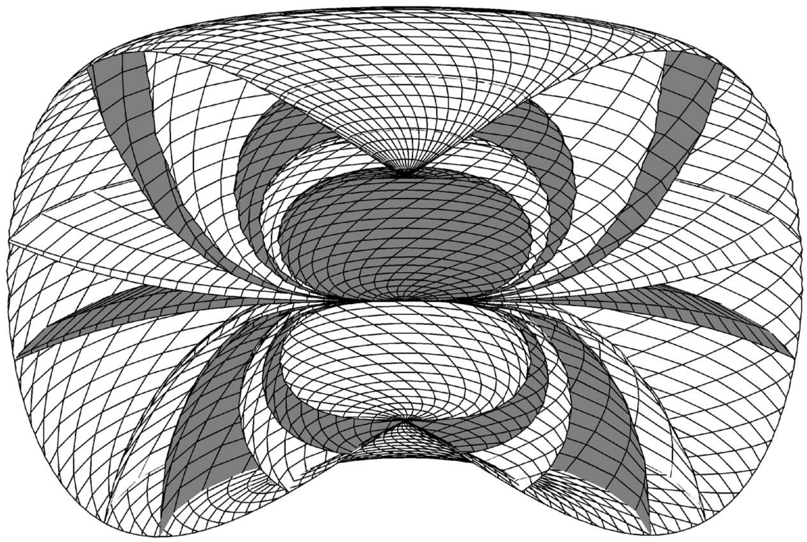

FIG. 3. A set of helicoidal trajectories in the subspaces generated by α_{2i-1}, α_{2i} and $\alpha_{2i-1,2i}$, as functions of time $t \in (0,1)$, ϕ and several fixed values of λ_i inside the unit sub-Riemannian sphere.

$$\|\Lambda x + \dot{x}_0\|^2 = \|\dot{x}_0\|^2, \tag{18}$$

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

Proof: The equation $\dot{x} = \exp(t\Lambda)\dot{x}_0$ is equivalent to $\Lambda\dot{x} = \exp(t\Lambda)\Lambda\dot{x}_0$. By integrating the latter in the interval $[0,t]$, we obtain

$$\Lambda x(t) = (\exp(t\Lambda) - \mathrm{I})\dot{x}_0 = \dot{x}(t) - \dot{x}_0,$$

that corresponds to Eq. (17) for $k=1$. A standard induction argument finishes the proof of the first result, the second is immediate. □

Corollary: The end points of extremal curves at fixed time in the three dimensional subspaces $\mathrm{span}\{\alpha_{2i-1}, \alpha_{2i}, \alpha_{2i-1} \wedge \alpha_{2i}\}$ *are given by the family of curves*

$$-\tan\left(\frac{a_i^2 - a_i(\bar{x}_{2i-1}\cos\phi_i - \bar{x}_{2i}\sin\phi_i)}{2\bar{z}_{2i-1,2i}}\right) = \frac{\bar{x}_{2i-1}\sin\phi_i + \bar{x}_{2i}\cos\phi_i}{\bar{x}_{2i-1}\cos\phi_i - \bar{x}_{2i}\sin\phi_i}. \tag{19}$$

These curves project down to the planes $\mathrm{span}(\alpha_{2i-1}, \alpha_{2i})$ *to the family of cochleoids*

$$\rho_i^2 = \bar{x}_{2i-1}^2 + \bar{x}_{2i}^2 = \frac{-2a_i}{\lambda_i t}(\bar{x}_{2i-1}\sin\phi_i + \bar{x}_{2i}\cos\phi_i), \tag{20}$$

for fixed ϕ_i, with $a_i = tR_i$. The set of intersection points of the above three dimensional curves with the plane $\bar{x}_{2i}=0$ is the curve

$$\bar{x}_{2i-1} = \frac{2a_i}{\lambda_i t}\sin\frac{\lambda_i t}{2}, \quad \bar{z}_{2i,2i-1} = \frac{a_i^2}{\lambda_i t} - \frac{a_i^2\sin\lambda_i t}{\lambda_i^2 t^2}. \tag{21}$$

Proof: The first two results follow from the last theorem, which leads to

$$-\lambda_i t = 2\tan^{-1}\frac{\bar{x}_{2i-1}\sin\phi_i + \bar{x}_{2i}\cos\phi_i}{\bar{x}_{2i-1}\cos\phi_i - \bar{x}_{2i}\sin\phi_i}. \tag{22}$$

The last plane curve arises after observing that the plane $\bar{x}_{2i}=0$ is obtained after setting $\phi_i = -\lambda_i t/2$. □

Remarks: Remember that the cochleoid is the barycenter of the circle. Since

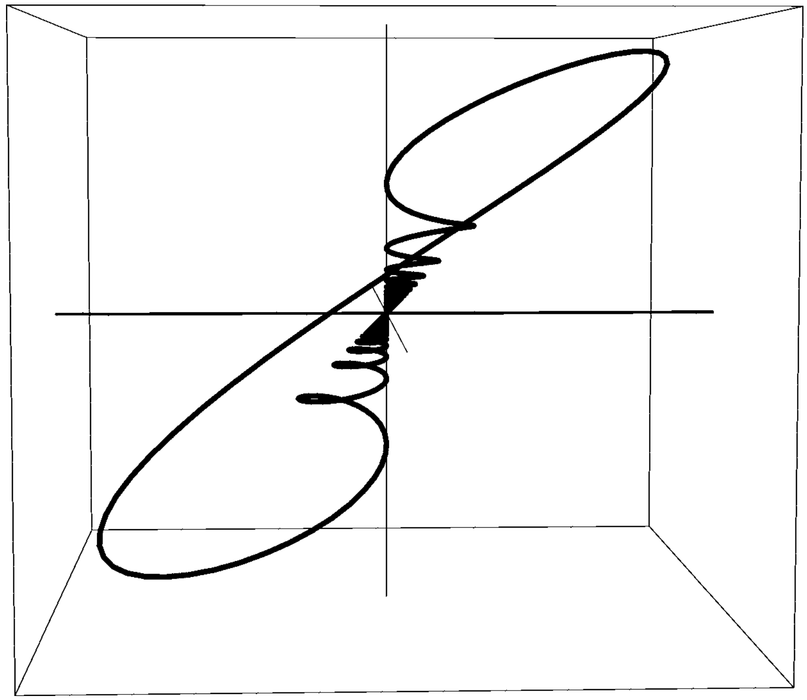

FIG. 4. The end points of trajectories of fixed length in the subspace span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i}\}$, as functions of the frequency λ_i for $t=1$ and $\phi_i=0$.

$$\dot{\bar{z}}_{2i,2i-1} = \frac{-a_i}{t}(\bar{x}_{2i-1}\sin\phi_i + \bar{x}_{2i}\cos\phi_i),$$

the inverses of the multipliers λ_i give an amount of how much the horizontal trajectories deviate from the directions parallel to the planes span$\{\alpha_{2i-1},\alpha_{2i}\}$. *In particular, for $t=2\pi k/\lambda_i$, for integral k, $\dot{z}_{2i,2i-1}$ vanishes, whereas it reaches its maximum absolute value $2R_i^2/\lambda_i$ for the times $t=(2k+1)\pi/\lambda_i$ (see Fig. 2). Finally, let us remark that since*

$$\rho_i = \frac{2R_i}{\lambda_i}\sin\frac{\lambda_i t}{2}, \quad t>0 \tag{23}$$

does not depend on ϕ_i, then the resulting curves, for λ_i and t constant as functions of ϕ_i, are circles of radii ρ_i centered at the origin and parallel to the plane span$\{\alpha_{2i-1},\alpha_{2i}\}$. *For $t=2k\pi/\lambda_i$, the radii are zero and for $t=(2k-1)\pi/\lambda_i$ they acquire their maxima $2R_i/|\lambda_i|$.*

In Fig. 4, the locus of end points $(\bar{x}_{2i-1},\bar{x}_{2i},\bar{z}_{2i,2i-1})$ of trajectories starting at the origin for $\phi_i=0$ and $t=1$ is shown. The displayed curve met the (vertical) $\bar{z}_{2i,2i-1}$-axis at $\lambda_i=2\pi k_i$, for $k_i=\pm 1,\pm 2,\ldots$. The projection of this curve on the plane span$(\alpha_{2i-1},\alpha_{2i})$ is the cochleoid displayed in Fig. 5. Here the origin is met again at $\lambda_i=2\pi k_i$, for $k_i=\pm 1,\pm 2,\ldots$. The projection on the plane span$(\alpha_{2i-1},\alpha_{2i}\wedge\alpha_{2i-1})$ is the first curve of Fig. 6, the other three curves correspond to $\phi_i=\pi/4,\pi/2$, and $3\pi/4$, respectively. The intersection of all the curves of the kind given in Fig. 4 with the plane $\bar{x}_{2i}=0$ is given in Fig. 7.

Let $[0,T]$ be a sufficiently small interval, and let $t\mapsto((x,z),(u,\Lambda))$ be a trajectory of the system, satisfying the initial condition $((0,\mathbf{0}),(\dot{x}_0,\Lambda))$. The exponential mapping projects the covector $\xi=(\dot{x}_0,\Lambda)$ into the geodesic arc (x,z). The study of the geometric properties of the exponential mapping passes then through an appropriate parametrization of the covector in terms of algebraic invariants of the problem. The first component of the covector can be parametrized according to Eqs. (4) and (5).

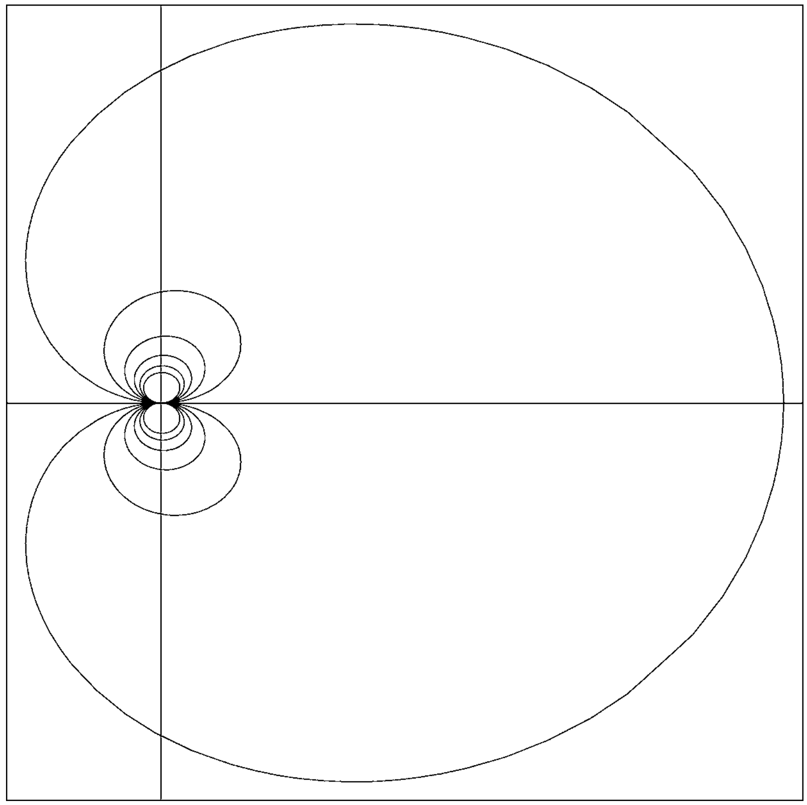

FIG. 5. The projection on the plane span$\{\alpha_{2i-1}, \alpha_{2i}\}$ for $\phi_i=0$ as function of the frequency λ_i.

In the coordinate system given by $\{\alpha_1, \alpha_2, \ldots, \alpha_{\lfloor n/2 \rfloor}\}$, Λ is block diagonal and therefore it is parametrized by its eigenvalues. Set $\alpha=Re$, with an orthogonal matrix R, then $R\Lambda R^T$ is block diagonal in the standard base $\{e_i | e_i=(0, \ldots, 1, \ldots, 0)\}$. Thus, the exponential map, $\mathbb{R}^{3n/2} \mapsto \mathbb{R}^{3n/2}$, is fully parametrized as follows.

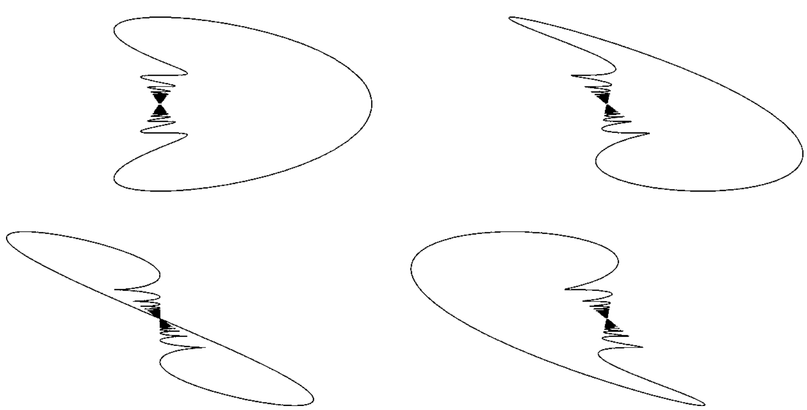

FIG. 6. The projection on the plane span$\{\alpha_{2i-1}, \alpha_{2i-1} \wedge \alpha_{2i}\}$ for four values of ϕ_i.

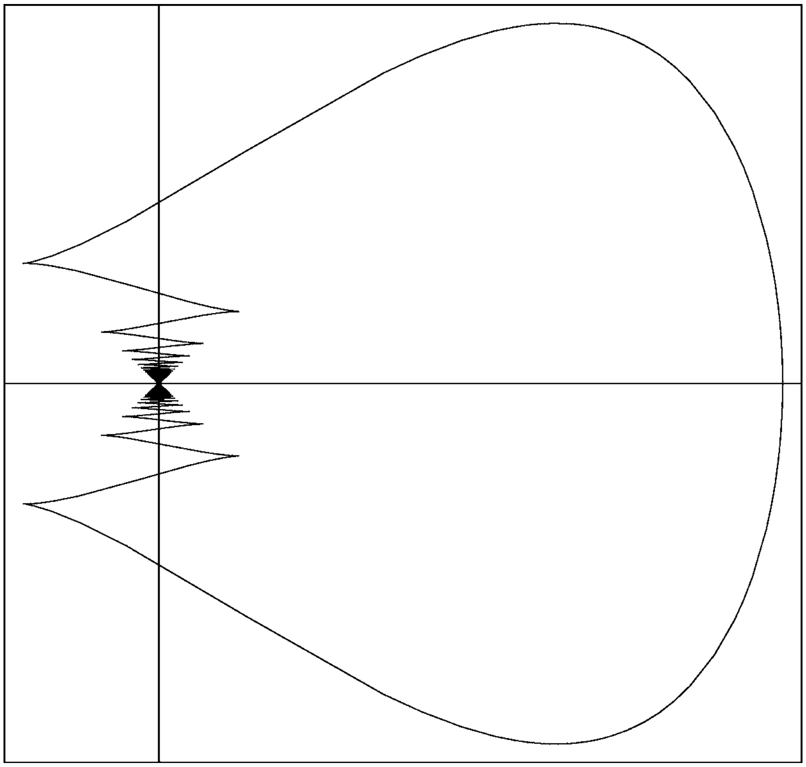

FIG. 7. The intersection of the curves for all ϕ_i and λ_i with the plane $\bar{x}_{2i}=0$.

$$(\underbrace{R_1,\phi_1,\lambda_1},\underbrace{R_2,\phi_2,\lambda_2},\ldots,\underbrace{R_{n-2/2},\phi_{n-2/2},\lambda_{n-2/2}},\underbrace{R_{n/2},\phi_{n/2},\lambda_{n/2}}) \mapsto (x;\bar{z}_{1,2},\bar{z}_{3,4},\ldots,\bar{z}_{n-1,n}),$$

$$(\underbrace{R_1,\phi_1,\lambda_1},\underbrace{R_2,\phi_2,\lambda_2},\ldots,\underbrace{R_{\lfloor n-2/2\rfloor},\phi_{\lfloor n-2/2\rfloor},\lambda_{\lfloor n-2/2\rfloor}},\underbrace{R_{\lfloor n/2\rfloor},\phi_{\lfloor n/2\rfloor},\lambda_{\lfloor n/2\rfloor}},R_n) \mapsto (x;\bar{z}_{1,2},\bar{z}_{3,4}\ldots,\bar{z}_{n-2,n-1}),$$

for n even and n odd respectively.

VI. SMALL RADII SPHERES AND WAVE FRONTS

The wave front is defined as the set of end points of geodesics of fixed length. The unit sphere is the set of points of geodesics at unit sub-Riemannian distance from the origin and is contained in the wave front. A set of examples is given in arc length units.

In Fig. 8, the wave front of trajectories in the three dimensional subspaces span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i})$ is shown. For a given eigenvalue λ_i and, say, $t=1$, the inscribed paraboloids are parametrized by the initial velocities, i.e., by their moduli $R_i\in[0,1)$ and their orientation $\phi_i\in[0,2\pi]$. The flat section corresponds to rectilinear trajectories for $\lambda_i=0$ and all surfaces lie inside the unit sub-Riemannian sphere. For $\lambda_i=2\pi$, the paraboloid degenerates into a segment on the $\bar{z}_{2i,2i-1}$ axis from 0 to 1.

A well known surface of revolution in three dimensional ambient subspaces corresponds to the small unit Heisenberg–Brockett sphere and the wave front in span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i}\}$ for $t=1$, as functions of λ_i and ϕ_i starting from the one forms used here. In Fig. 9, the sphere and part of the wave front inside it are shown for $\pi/2<\phi_i<2\pi$. This surface of revolution can be obtained by a rotation of the curve shown in Fig. 4. The cochleoidal helices given by Eq. (19) are easily recognized as well as the cochleoids of Eq. (20). Finally, the rotation around the axis $\alpha_{2i}\wedge\alpha_{2i-1}$ of the curve given by Eq. (21) yields the wave front shown in Fig. 10. The unit sphere for other equivalent one forms associated with $\dot{z}_{2i,2i-1}=x_{2i-1}\dot{x}_{2i}-x_{2i}\dot{x}_{2i-1}+a(d/dt)x_{2i-1}x_{2i}$, for several values of the real constant a shown in Fig. 11. The wave front changes correspondingly.

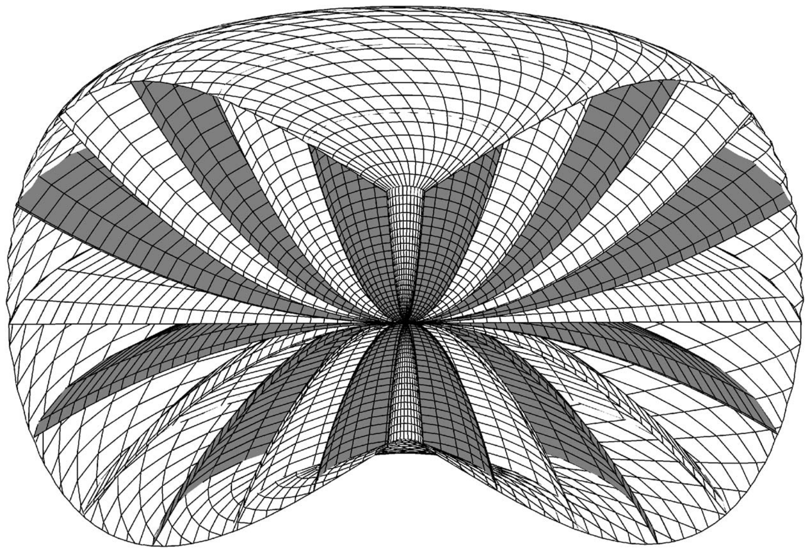

FIG. 8. A three dimensional section of the wave front in span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i}\}$, shown inside the unit sub-Riemannian sphere.

New examples of sections of the wave front and the unit sphere are shown in Fig. 12 in span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_0\}$, span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i}\wedge\alpha_0\}$, for n odd, for $\phi_i=0$.

VII. CONJUGATE LOCUS AND GEODESICS LENGTHS

From the above sections, we have gained some insight on the horizontal curves joining the origin with the axes $x_f=0$. Let us now find the conjugate locus of the origin, that is, the points for

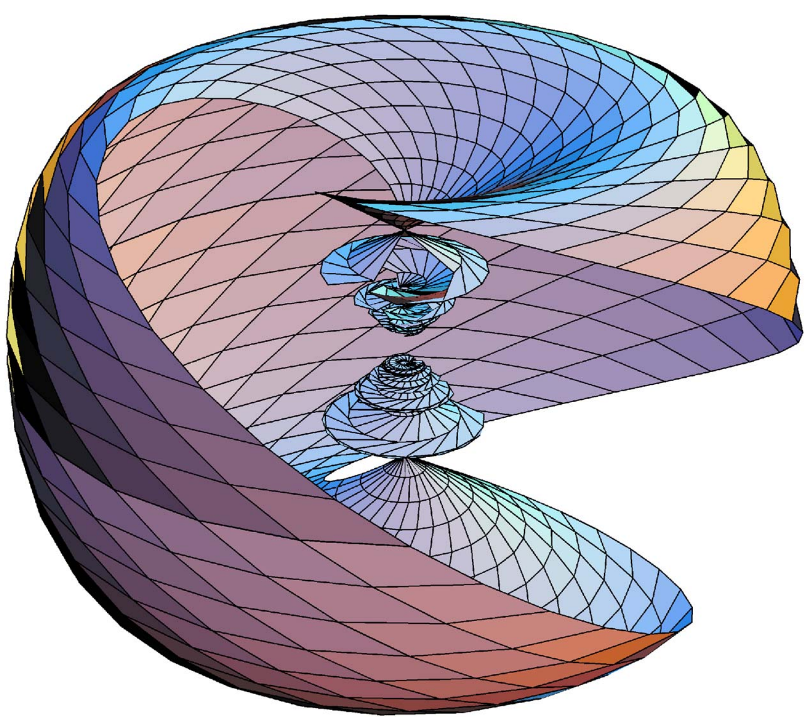

FIG. 9. (Color online) The wave front in span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i}\}$, as function of λ_i and ϕ_i.

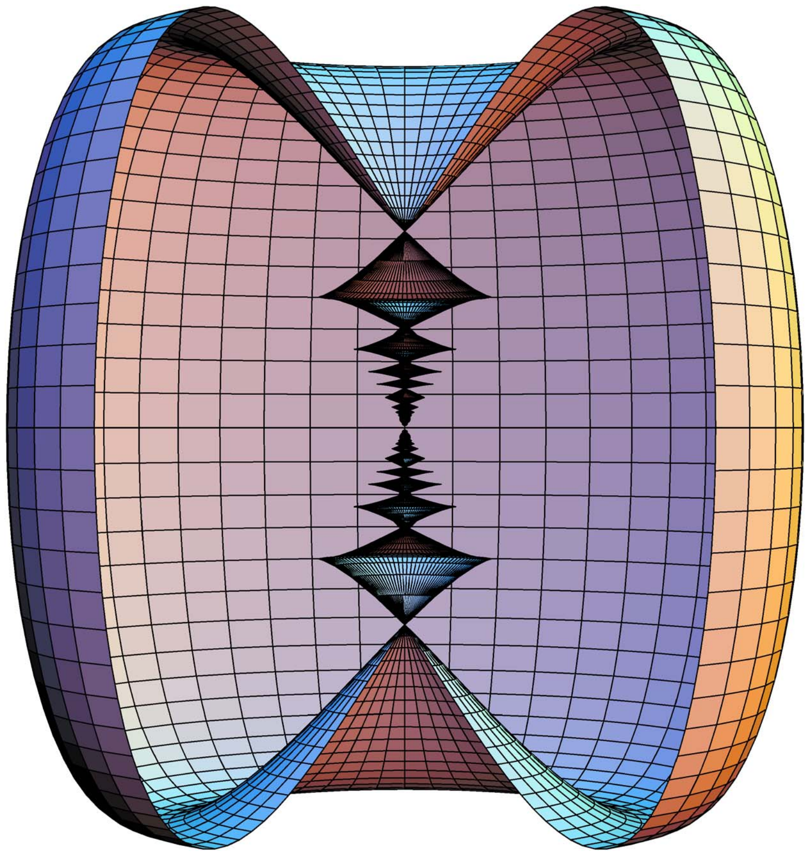

FIG. 10. (Color online) The wave front as in Fig. 9, but obtained by a rotation of Fig. 7.

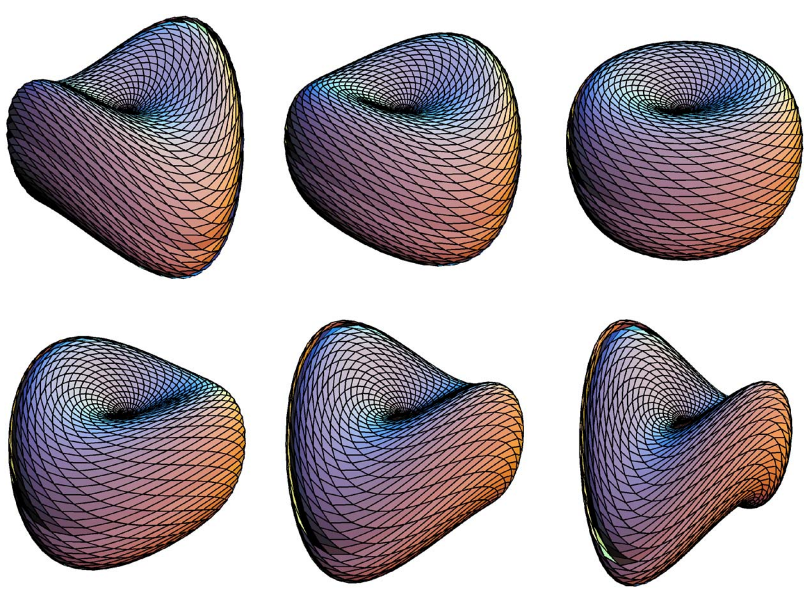

FIG. 11. (Color online) The unit sphere for several gauges.

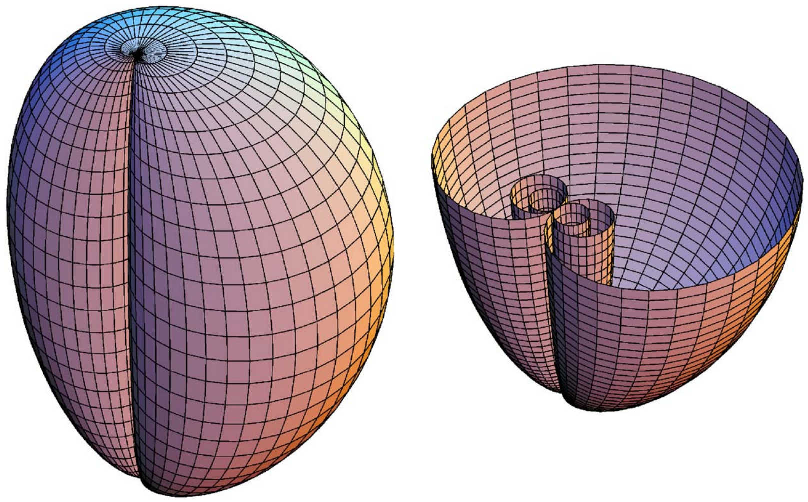

FIG. 12. (Color online) Sections of the unit sphere and the corresponding wave front for n odd, in span$\{\alpha_{2i-1},\alpha_{2i},\alpha_0\}$ for $\phi_i=0$.

which the exponential mapping fails to be an immersion. For this end, the standard procedure is to compute the set of zeros of the corresponding Jacobian. In our problem the Jacobian is, for n even,

$$\mathrm{Det}(\exp) = \prod_{i=1}^{\lfloor n/2\rfloor} \frac{2R_i^3 t}{\lambda_i^4}(2\cos\lambda_i t + \lambda_i t \sin\lambda_i t - 2),$$

which is zero for $t=0$, and for all $\lambda_j t = 2\pi k_j$, with k_j integer. The first zero of the Jacobian after the origin is at the largest λ_i, say, $\lambda_{\max}$, such that $\lambda_{\max}t=2\pi$. Therefore, the exponential map is singular along the axes $\alpha_{2i-1}\wedge\alpha_{2i}$, for n even. Also for n odd follows the same result if additionally $R_{\lfloor n/2\rfloor+1}=0$, since the Jacobian for the nontrivial coordinates is the same as for n even. The given axes are the conjugate locus of the origin and the first conjugate point of the origin is reached at time $2\pi/\lambda_{\max}$.

An instructive way to recognize the conjugate locus of the origin is to consider the Jacobi fields[7,12] associated with the Lagrangian equation (7) defined by the second variations

$$J(t) = \left. \frac{\partial}{\partial\tau} q_\tau(t) \right|_0 ,$$

where τ is a parameter associated with a family of geodesics $q_\tau(t)$. In the subspaces span$\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i}\wedge\alpha_{2i-1}\}$, Jacobi fields can be calculated by considering the variation of the angle ϕ_i. Since $z_{2i,2i-1}$ does not depend on ϕ_i, the resulting vector fields are parallel to the plane span$\{\alpha_{2i-1},\alpha_{2i}\}$ and are associated with the rotations $x_{2i-1}\partial_{2i}-x_{2i}\partial_{2i-1}$. Therefore, the Jacobi fields vanish for $x=0$, making again clear that the conjugate locus are the axes with directions $\alpha_{2i}\wedge\alpha_{2i-1}$. For n odd, it is needed again that $R_{\lfloor n/2\rfloor+1}=0$. These fields satisfy the Jacobi equations, which in our case are just $\ddot{J}=\Lambda\dot{J}$. In the figures of the trajectories of Sec. V, the integral lines of the Jacobi fields can be easily recognized as the circles of radii ρ_i given by Eq. (23) and centered around the axes with coordinates

$$z_{2i,2i-1} = \frac{2R_i^2}{\lambda_i^2}\arcsin\left(\frac{\lambda_i\rho_i}{2R_i}\right) - \frac{R_i\rho_i}{\lambda_i}\left(1-\frac{\lambda_i^2\rho_i^2}{4R_i^2}\right)^{1/2}.$$

The radii become zero as the trajectories approach the conjugate locus at $z_{2i,2i-1}=t_f^2R_i^2/2\pi k_i$. However, for other gauges the $z_{2i,2i-1}$ depend on ϕ_i, as can be recognized in Fig. 11. In Fig. 13, the Jacobi field is shown schematically in span$\{\alpha_{2i-1},\alpha_{2i}\}$.

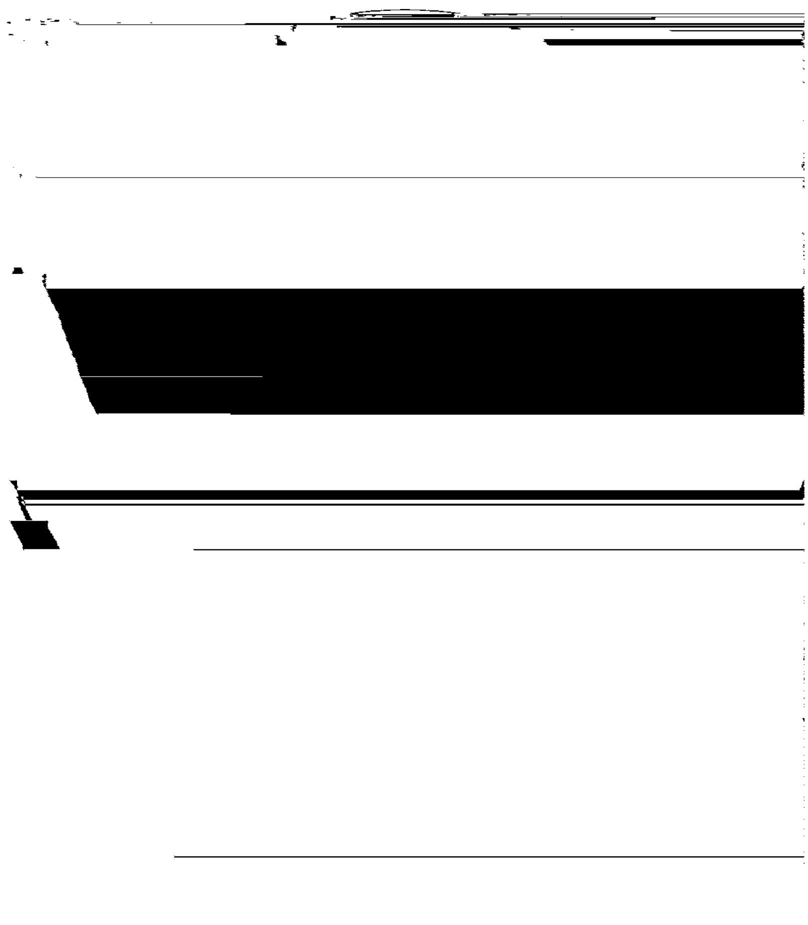

FIG. 13. The Jacobi field on span $\{\alpha_{2i-1},\alpha_{2i}\}$ vanishes at the origin of the trajectory (broad line), given by the intersection with the thin circle (given here only as a visual aid).

A point x_c is called a cut point along a geodesic going from x_0 to x_t, if the geodesic ceases to be minimizing after x_c, i.e., the length of the geodesic is greater than the distance between x_0 and x_t. A well known result[10] states that if x_c is the cut point of x_0 along a geodesic $\gamma=x(t)$, $0\leqslant t<\infty$, then at least one of the following statements holds: (i) x_c is the first conjugate point of x_0 along γ; (ii) there exist at least two minimizing geodesics from x_0 to x_c. In the step-2 case under consideration, both statements hold since at the first conjugate point an infinite number of trajectories coincide.

A particularly interesting case are those geodesics joining the origin $(x_0,z_0)=(0,0)$ at $t=0$ with the point $(0,z_f)$, for n even (and $R_{[n/2]+1}=0$ for n odd), at $t=t_f>0$, for certain $z_f=z(t_f)$ not fixed but given by the constraints. Clearly these are (circular) loops in span$\{\alpha_{2i-1},\alpha_{2i-1}\}$.

Proposition: For n even and commensurable frequencies $\lambda_j=2\pi k_j/t_f$, *with* k_j *being nonzero integers, there is an infinite number of trajectories starting at the origin* $(x_0,z_0)=(0,0)$ *and ending at the axis* $(x_f,z_f)=(0,z_f)$. *Further,*

$$z_f=\sum_{j=1}^{[n/2]}\zeta_j\alpha_{2j}\wedge\alpha_{2j-1},$$

where the $\pm i\zeta_j=\pm it_fR_j^2/\lambda_j$ *are the nonzero eigenvalues of* z_f. *For n odd, the same holds if* $R_{[n/2]+1}=0$.

Proof: From the theorem of Sec. V, to have $x_f=0$, for n even. It is necessary that $t_f\lambda_i$ are all integer multiples of 2π, or for n odd if additionally $R_{\&\#63728;n/2\&\#63739;+1}=0$ (i.e., $\bar{x}_n=0$, for all times). Then, all frequencies are integer multiples of the frequency $2\pi/t_f$, say, $\lambda_i=k_i 2\pi/t_f$, for $\{k_i, i=1,\ldots,n/2\}$, a set of nonzero integers. In that case, since the initial speeds $\dot{x}_0$ are arbitrary [with $(\dot{x}_0)_n=0$ for n odd], there are infinitely many geodesics joining both points. It is simple to recognize in the aforementioned theorem that only the matrix elements $(z_f)_{2j-1,2j}=t_f R_j^2/\lambda_j$ are nonzero, where the subindices denote the component in the direction $\alpha_{2j}\wedge\alpha_{2j-1}$. Therefore, for the even case the quantities $\pm i(z_f)_{2j,2j-1}$ are the eigenvalues of the matrix z_f. In the odd case, there is an additional zero eigenvalue. □

In Fig. 4, showing a curve on the wave front in the subspace $\text{span}\{\alpha_{2i-1},\alpha_{2i},\alpha_{2i-1}\wedge\alpha_{2i}\}$ the above results for z_f can be interpreted as a subset of the set of points for which the curve touches the vertical axis. As we can see, this axis is met at an infinite discrete set of points. The first intersection (the first conjugated point of the origin) of the unit sphere with the vertical axis occurs for trajectories with frequency $2\pi\lambda_i$, the second for frequency $4\pi\lambda_i$, which means that in $\text{span}\{\alpha_{2i-1},\alpha_{2i}\}$ the circular trajectories are swept twice, and so on. Clearly, there are conjugate points of the origin arbitrarily near from itself.

VIII. CONCLUSIONS

In this work, we study a class of dynamical systems given in terms of distributions of real analytic vector fields. We analyze in detail the problem of minimization of kinetic energy for solution curves of differential systems defined by real polynomial vector fields. For the nilpotent Lie algebras associated with this problem, a Philip Hall basis is explicitly exhibited. The step-2 case is discussed in detail, the normal trajectories are explicitly written, and some commented pictured are presented. The sub-Riemannian sphere, the conjugate locus, and the wave front are calculated for some subspaces.

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