

Phasors

What are Phasors?

All voltages and currents in a linear system at ac (all sinusoids) can be written in the form:

$$v(t) = V_{\max} \cos(\omega t + \phi)$$

So far, we have needed to use differential equations to analyse circuits containing reactive components (capacitors and inductors). This method is time consuming, mathematically difficult and prone to errors.

Thankfully, there is an easy alternative. It takes advantage of the facts that once the initial transients settle down after switching a system on...

- 1) Every voltage and current in a linear ac system is just another sinusoid.
- 2) The frequency ω is the same everywhere in the system.

So the new method takes these factors for granted and just looks at the amplitude (rms voltage) and the phase of the wave (i.e. how far through the cycle before $t=0$ was the nearest peak?). We call the resulting description of the circuit "*PHASOR NOTATION*". The method gets rid of all differential equations and lets you analyse inductors and capacitors in the same way you would look at resistors (ohm's law, potential divider equation etc...).

How to use them...

- Draw out the circuit diagram as normal.
- Replace all the labels in the diagram with phasors as shown below.
- Treat all reactive components (L and C) as if they were just resistors and carry out normal circuit analysis (ohm's law, potential dividers etc...) to find your answer (in phasor form).
- Convert the answer back to time-domain as shown below.

Changing from time-domain to phasors

Voltage or Current Sources

Time domain value of a voltage (or current) is:

$$v(t) = V_{\max} \cos(\omega t + \phi)$$

Where V_{\max} is the peak value of the wave in each cycle and ϕ is the phase shift.

Make sure it's written in this form first. The method I describe **will not work** if there's a minus sign at the front or if it's a sine function rather than a cosine (other approaches are available for this purpose)! Remember that a sine wave is the same as a cosine wave delayed by quarter of a wavelength ($-\pi/2$ shift) and that a negative wave is the same as a positive one advanced by a half wavelength ($+\pi$ shift).

Here's an example of converting a waveform to the positive cosine form:

$$i(t) = -3\sin\left(75t - \frac{p}{4}\right)$$

Shift forward by half - wavelength to remove minus sign...

$$\Rightarrow i(t) = 3\sin\left(75t - \frac{p}{4} + p\right)$$

Shift backward by quarter wavelength to change sin to cos...

$$\Rightarrow i(t) = 3\cos\left(75t - \frac{p}{4} + p - \frac{p}{2}\right)$$

Tidy up to get the answer...

$$\Rightarrow i(t) = 3\cos\left(75t + \frac{p}{4}\right)$$

Now the voltage (or current) is in this form, it's easy to change into a phasor.

- We forget about the frequency.
- We talk about rms voltage of the wave instead of its peak voltage to allow our ac power calculations to work properly (you'll need it later!)
- A phasor is a complex number written as below...

$$\tilde{V} = V_{rms}[\cos \mathbf{f} + j \sin \mathbf{f}]$$

So for the example above we calculate the rms current,

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = \frac{3}{\sqrt{2}} = 2.12$$

We can now write it in phasor notation as:

$$\begin{aligned}\tilde{I} &= I_{rms}[\cos \mathbf{f} + j \sin \mathbf{f}] \\ &= 2.12 \left[\cos\left(\frac{p}{4}\right) + j \sin\left(\frac{p}{4}\right) \right] \\ \tilde{I} &= 1.5 + j1.5\end{aligned}$$

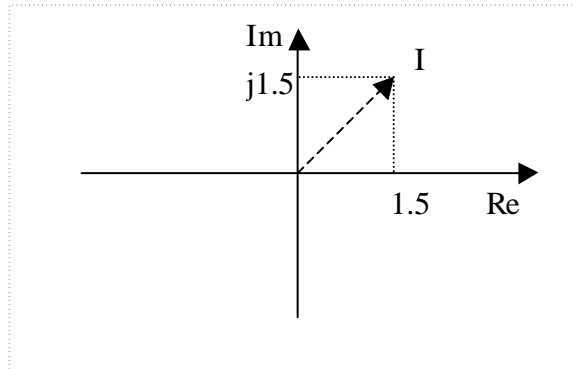
So it's a much nicer way to look at things than the full time-domain form. Note that this works for current sources in exactly the same way as voltage sources. We can also express phasors in exponential form:

$$\tilde{V} = V_{rms} e^{j\mathbf{f}} = V_{rms} \angle \mathbf{f}$$

Which for the example above is...

$$\tilde{I} = 2.12 e^{j\frac{p}{4}} = 2.12 \angle \frac{p}{4}$$

We can also draw them on a *phasor diagram*, which is just a drawing of what the phasor looks like as a complex number:



Capacitors

In the time domain, you can describe a capacitor as:

$$i(t) = C \frac{dv(t)}{dt}$$

Where C is the capacitance.

$$v(t) = V_{\max} \cos(\omega t + \phi)$$

Differentiate

$$\Rightarrow \frac{dv(t)}{dt} = -\omega V_{\max} \sin(\omega t + \phi)$$

Substitute into capacitor equation..

$$\Rightarrow i(t) = -\omega C V_{\max} \sin(\omega t + \phi)$$

Rearrange into positive cosine form...

$$i(t) = \omega C V_{\max} \cos\left(\omega t + \frac{\pi}{2} + \phi\right)$$

Transform voltage and current into phasor form...

$$\Rightarrow \tilde{V} = V_{rms} e^{j\phi}$$

$$\tilde{I} = \omega C V_{rms} e^{j\left(\frac{\pi}{2} + \phi\right)} = \omega C V_{rms} e^{j\frac{\pi}{2}} e^{j\phi} = j\omega C V_{rms} e^{j\phi}$$

Combining these equations. .

$$\tilde{I} = j\omega C \tilde{V}$$

Rearranging this...

$$\tilde{V} = \frac{1}{j\omega C} \tilde{I}$$

So it now looks a lot like a resistor ($v=RI$). Instead of thinking about its capacitance, we now talk about its *impedance*, ($Z=V/I$) and use it in the same way as a resistor. The impedance is then just:

$$Z = \frac{1}{j\omega C} = \left(-\frac{j}{\omega C}\right)$$

Inductors

For inductors, we can derive impedance too. Here,

$$Z = j\omega L$$

Resistors

The equation for a resistor is: $v(t) = R i(t)$.

When we use phasors, this becomes:

$$V = R I$$

So:

$$Z = R$$

So we treat resistors exactly the same when we're using phasor notation. The impedance exactly equals the resistance.

Impedance, Admittance, Reactance and Susceptance

We now have a few different ways of describing the I-V characteristics for a device...

Resistance

This is the ratio of magnitude of voltage and current: $R=v(t)/i(t)$

It only exists for a resistor as the differential equations for capacitors and inductors can't be changed into this form.

Conductance

This is the inverse of resistance: $g=i(t)/v(t)$

Again, it only exists for a resistor.

Impedance

This is similar to resistance, but is the ratio of the *phasor* voltage and current. $Z=V/I$

This means that it is able to show both the magnitude **and phase** relationship between voltage and current. This can be used to represent all of the passive circuit components.

Admittance

This is the inverse of impedance. $Y=I/V$

Reactance

This is a "baby version" of impedance. It just forgets about the j . It therefore tells you nothing about phase. It is given the symbol, X . It is only defined for components, which cause phase shifting (not resistors)

Susceptance

This is the inverse of reactance.

Summary

Here are the parameters for each of the three passive components...

	Resistor	Capacitor	Inductor
Resistance	R	-	-
Conductance	$1/R$	-	-
Impedance	R	$1/j\omega C$	$j\omega L$
Admittance	$1/R$	$j\omega C$	$1/j\omega L$
Reactance	-	$1/\omega C$	ωL
Susceptance	-	ωC	$1/\omega L$

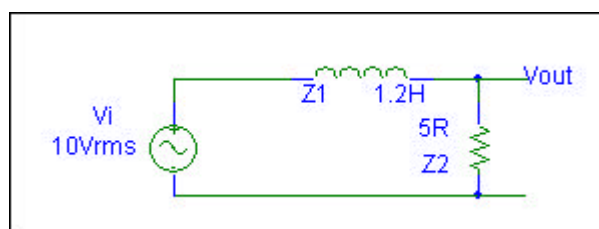
In fact, from the table you can see that you only need to *learn the impedance* and from that, you can work out all the rest.

When Phasors Turn Evil

Phasors are a very useful tool, but can lead to much grief if used unwisely! They will give wrong answers if you use them in a situation where the original assumptions are invalid. Here's a list of possible errors...

- Do not use them for DC signals, or even things that look like sinusoids such as saw-tooth, triangular or square waves. **Phasors work only for sinusoids!**
- Remember to convert signals to positive cosine waves first if you're using this method!
- Remember to use rms values NOT PEAK VALUES otherwise power calculations will be wrong!
- Phasors do not reveal anything about a system's transient response (unless combined with Laplace or Fourier Transforms which we'll meet later). Beware because for example you could design a "perfect" system to raise an elevator by ten metres using phasors and then find out that it takes ten years to get there or maybe that it bounces up and down over a height of 100m first!

Example



Find the output voltage for the circuit above, using phasor analysis at a frequency of 10rad/s.

Firstly, convert the circuit into phasor form...

The voltage source has an rms value of 10V.

The resistor has impedance 5Ω .

The inductor has impedance of $j\omega L = j(10 \times 1.2) = j12\Omega$.

Now find the output voltage...

Use the potential divider rule:

$$V_o = V_i \frac{Z_2}{Z_1 + Z_2} = V_i \frac{R}{R + j\omega L} = 10 \frac{5}{5 + j12} = 50 \left(\frac{5 - j12}{25 + 144} \right)$$

$$\Rightarrow V_o = (1.479 - j3.550) \text{V}$$

This can be rewritten as...

$$V_o = (3.85 \angle -1.18 \text{rad}) \text{V} = (3.85 \angle -67.4^\circ) \text{V}$$

We can now easily rewrite the phasor answer above as a positive cosine wave at 10rad/s with magnitude $3.85 \times \sqrt{2} = 5.44$ (to get peak voltage instead of rms) and phase -67.4° .

$$v_o(t) = 5.44 \cos(10t - 67.4^\circ) \text{V}$$

As we'll see, often there's no need to do this conversion back to a waveform, as it'll actually become easier to visualise things in the phasor domain!