Topology

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Abstract

These notes are a quick introduction to basic point-set topology. They are based on a three week course I conducted at MTTS 2003. Please send comments/corrections to a_habib@yahoo.com.

Contents

1 Metric Spaces

We assume you have already encountered metric spaces, and the basic notions regarding them. At the very least, you would be aware that the Euclidean space $\mathbb{R}^n$ is equipped with a notion of distance (or metric) between its points. The presence of this metric allows us to set up ideas of limit and continuity and to develop calculus. It is a remarkable fact that much of what is done with a metric can also be done in its absence, provided we adopt a certain viewpoint as to the meaning of "nearness" and related concepts. The task of this booklet is to give a quick introduction to this more general approach, which is called Topology.

We start with a quick revision of the basic facts about metric spaces. The chief motivating example is $\mathbb{R}^n$ with the Euclidean distance

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$
where $x = (x_i)$ and $y = (y_i)$.

We extract the most commonly used properties of the Euclidean distance formula to formulate the abstract definition of a metric space:

Definition 1.1 A metric space (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ with the following properties:

1. (Positivity) $d(x, y) \geq 0 \quad \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$.
2. (Symmetry) $d(x, y) = d(y, x) \quad \forall x, y \in X$.
3. (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$.

The function d is called a metric on X.

Since the definition of metric spaces is based on properties of Euclidean space, we can use our spatial intuition when working with them. Questions involving limits can now be posed and answered for such spaces.

Example 1.2 Any set X with the *discrete metric*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

is called a discrete metric space. □

Example 1.3 $X = \mathbb{R}^n$ with

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

is a metric space. □

Example 1.4 $X = \mathbb{R}^n$ with

$$d_\infty(x,y) = \max_i |x_i - y_i|$$

is a metric space. □

Example 1.5 Let $X = C[0,1]$, the real-valued continuous functions from $[0,1]$. It can be given the *uniform metric*,

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

□

Example 1.6 Let $X = \{(a_i) : a_i \in \mathbb{N}\}$, the set of sequences of natural numbers. The following is a metric on X:

$$d(x,y) = \begin{cases} 0 & x = y \\ \dfrac{1}{r} & x_i = y_i \ \forall i < r \text{ and } x_r \neq y_r \end{cases}$$

□

Example 1.7 If (X,d) is a metric space, so are $(X, \min(1,d))$ and $(X, \dfrac{d}{1+d})$.
□

Exercise 1.8 Verify that all the functions defined in the above examples are actually metrics.

Definition 1.9 An *open ball* with centre x and radius r in a metric space is a set of the type

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

A subset of X is called *open* if it is a union of open balls.

Exercise 1.10 Consider $\mathbb{R}^2$ with the metrics d_2, d_1 and d_∞ defined earler. Sketch, for each of these metrics, the open balls with center at origin and radius R.

Almost every important notion involving metric spaces can be expressed in terms of open sets rather than through the metric itself. First, recall that if (x_n) is a sequence in a metric space (X, d) then we say it has a limit x ($\lim x_n = x$) if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \varepsilon$. This is equivalent to: For every open set U containing x, there is $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$.

Similarly, we usually first define continuity of a function $f : X \to Y$ between metric spaces by the requirement that $\lim x_n = x$ implies $\lim f(x_n) = f(x)$. And then we find that this is equivalent to demanding that for every open set U in Y, $f^{-1}(U)$ is open in X.

Exercise 1.11 Is the requirement that $f : X \to Y$ be continuous at the point $x \in X$ equivalent to demanding that for every open U containing $f(x)$, $f^{-1}(U)$ is open in X?

Also, many notions such as compactness and connectedness are defined directly in terms of open sets and not through the metric. Thus, open sets come to the fore and the metric's role is mostly relegated to providing the open sets (via the open balls).

Example 1.12 Consider $\mathbb{R}^2$ with the metrics d_1, d_2, d_∞. Observe that if x_n converges to x with respect to one of these metrics, it also converges to x with respect to the other ones. Alternately, note that although the three metrics have different open balls, they have the same open sets! Therefore they create the same definitions of limit, continuity, etc., and are essentially indistinguishable. $\square$

Exercise 1.13 Let (X, d) be any metric space. Define a new metric on X by $d' = d/(1 + d)$. Show that d and d' create the same open sets.

Definition 1.14 Let (X, d) be a metric space. The set $\mathcal{T}$ of open subsets of X is called the *topology* of X.

The key properties of $\mathcal{T}$ are:

1. $\emptyset, X \in \mathcal{T}$.
2. $\mathcal{T}$ is closed under arbitrary unions.
3. $\mathcal{T}$ is closed under finite intersections.
4. (Hausdorff Property) If $x, y \in X$ are distinct, then $\exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Properties 3 and 4 need the triangle inequality – in proofs, they replace it.[1]

2 Topological Spaces

We have identified the key properties of open sets in metric spaces. Now we abstract these properties into a definition.

Notation. The power set $\wp(X)$ consists of all the subsets of X.

Definition 2.1 A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subset \wp(X)$ such that

1. $\emptyset, X \in \mathcal{T}$.
2. $\mathcal{T}$ is closed under arbitrary unions.
3. $\mathcal{T}$ is closed under finite intersections.

The set $\mathcal{T}$ is called the *topology* of X and its members are called *open sets.*

Note that we have not included the Hausdorff condition. When a topological space satisfies this as well, we call it a *Hausdorff space.* A topology may arise as the open sets of a metric space but it need not.

Example 2.2 $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. The Hausdorff property fails to hold, and so $\mathcal{T}$ can't arise from a metric. □

Example 2.3 We list some simple examples of topological spaces. Start with any set X. Define a topology on it in one of the following ways.

1. $\mathcal{T}_{\text{dis}} = \wp(X)$, the *discrete topology.*
2. $\mathcal{T}_{\text{ind}} = \{\emptyset, X\}$, the *indiscrete topology.*
3. $\mathcal{T}_{\text{cof}} = \{U : U^c \text{ is finite or } X\}$, the *cofinite topology.*
4. $\mathcal{T}_{\text{coc}} = \{U : U^c \text{ is countable or } X\}$, the *cocountable topology.* □

Exercise 2.4 Verify that the collections given in the last example are actually topologies.

[1] Frechet defined metric spaces and focused on sequences. Hausdorff abstracted the four properties above, and focused on open sets.

These examples are useful for testing hypotheses because $\mathcal{T}$ is very explicitly given. Usually, $\mathcal{T}$ will be too large or chaotic to be described in such a simple manner.

Exercise 2.5 Consider a set X and choose two topologies for it from the list in the Example 2.3. Under what conditions on X will they be the same? (Answer this for each possible pair of topologies.)

Exercise 2.6 Which of the topologies in Example 2.3 are Hausdorff?

Exercise 2.7 Let A be an index set and $\{\mathcal{T}_\alpha : \alpha \in A\}$ a collection of topologies on X. Show that $\cap_\alpha \mathcal{T}_\alpha$ is a topology on X. What about $\cup_\alpha \mathcal{T}_\alpha$?

Definition 2.8 Let $\mathcal{T}_1, T_2$ be two topologies on X. If $\mathcal{T}_1 \subset T_2$ we say that T_2 is *finer*, or that T_1 is *coarser*. If $T_1 \subsetneq T_2$, we use the terms *strictly finer* or *strictly coarser*.[2]

Definition 2.9 Let $S \subset \wp(X)$. Let $\mathcal{T}_S$ be the coarsest topology on X which contains S. Then we call $\mathcal{T}_S$ the *topology generated by* S.

Exercise 2.10 Why does $\mathcal{T}_S$ exist? Is it unique?

Exercise 2.11 Consider $X = \mathbb{R}$.

1. Let $S = \{(a,b) : a < b\}$. Show $\mathcal{T}_S$ is the same as the topology of the metric $d(x,y) = |x-y|$. This is called the *standard topology* of $\mathbb{R}$ and we will denote it by $\mathcal{T}_d$.

2. Let $S' = \{(a,b) : a,b \in \mathbb{Q} \text{ and } a < b\}$, and $S'' = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a,\infty) : a \in \mathbb{R}\}$. Compare $\mathcal{T}_{S'}$ and $\mathcal{T}_{S''}$ with each other and with $\mathcal{T}_d$.

Example 2.12 Let $X = \mathbb{R}$ and $L = \{[a,b) : a < b\}$. Then $\mathcal{T}_L$ is called the *lower limit topology* of $\mathbb{R}$. Let us compare $\mathcal{T}_L$ with the standard topology $\mathcal{T}_d$ and the discrete topology $\mathcal{T}_{\text{dis}}$. It is easy to show that $\mathcal{T}_d \subsetneq \mathcal{T}_L \subset \mathcal{T}_{\text{dis}}$. To find out whether $\mathcal{T}_L = \mathcal{T}_{\text{dis}}$ we must see whether or not the singletons are in $\mathcal{T}_L$. For this we need information on how general members of $\mathcal{T}_L$ look. We give a general answer below. □

Exercise 2.13 Let $S \subset \wp(X)$. Then the topology generated by S is

$$\mathcal{T}_S = \left\{ \bigcup_{\alpha \in A} \Big(\bigcap_{j \in I_\alpha} U_j^\alpha \Big) : \begin{array}{l} A \text{ is an arbitrary index set,} \\ \text{each } I_\alpha \text{ is a finite index set,} \\ \text{and each } U_j^\alpha \in S. \end{array} \right\}.$$

[2]Other terminology for this is *weak* for coarse and *strong* for fine.

Exercise 2.14 Show that singletons are not open in the lower limit topology on $\mathbb{R}$. (Hence $\mathcal{T}_L \subsetneq \mathcal{T}_{\text{dis}}$)

Definition 2.15 Let $(X, \mathcal{T})$ be a topological space and $S \subset \wp(X)$ such that $\mathcal{T} = \mathcal{T}_S$. Then we call S a *subbasis* (or *subbase*) of $\mathcal{T}$. Members of S are called *subbasic sets.*

For example, a subbase for $\mathcal{T}_{\text{cof}}$ is $S = \{X \setminus \{a\} : a \in X\}$.

Notice that many times we only needed to take arbitrary unions to reach $\mathcal{T}_S$ from S, skipping the finite intersections step. Analyzing when this happens, we are led to the next definition.

Definition 2.16 $S \subset \wp(X)$ is a *basis* (or *base*) if finite intersections of members of S are also unions of elements of S. If S is a basis and $\mathcal{T} = \mathcal{T}_S$, we say S is a *basis of* $\mathcal{T}$.

Now we do a small example to show that topologies are *created* for a *purpose.*

Definition 2.17 A sequence (x_n) in a topological space X has limit x, denoted by $x = \lim x_n$, if for every open set U containing x, $\exists N \in \mathbb{N}$ such that $x_n \in U$ for every $n > N$.

Example 2.18 Consider $\mathbb{N}$ with the discrete topology. We often write $\lim x_n = \infty$ for a sequence (x_n) in $\mathbb{N}$, qualifying this as just "notation" since $\mathbb{N}$ has no element called ∞. But by an appropriate choice of topology, we can make this a regular equality. First, define

$$\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$$

Consider the basis

$$B = \{\{x\} : x \in \mathbb{N}\} \cup \{\{x, x+1, \ldots, \infty\} : x \in \mathbb{N}\}$$

Consider $\mathbb{N}_\infty$ with the topology generated by B. Now for a sequence (x_n) in $\mathbb{N} \subset \mathbb{N}_\infty$ the statement $\lim x_n = x$ has the standard meanings, both for $x \in \mathbb{N}$ and for $x = \infty$. □

We often have to deal with the open sets containing a certain fixed point. If U is an open set containing x, we call U a neighbourhood of x.

Definition 2.19 Let $(X, \mathcal{T})$ be a topological space and $x \in X$. $\mathcal{U} \subset \mathcal{T}$ is a *basis at* x, if each $U \in \mathcal{U}$ is a neighbourhood of x and for every open set O containing x there is a $U \in \mathcal{U}$ such that $U \subset O$.

Exercise 2.20 Give bases at $0 \in \mathbb{R}$ for the standard and lower limit topologies.

Notation From now on a topological space $(X, \mathcal{T})$ will be referred to by just X.

3 Continuous Functions

Let X, Y be topological spaces.

Definition 3.1 A function $f : X \to Y$ is *continuous at* $x \in X$ if for every open V containing $f(x)$, there is an open U containing x such that $f(U) \subset V$. If f is continuous at every $x \in X$, we just say that f is continuous.

Exercise 3.2 $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open whenever V is an open subset of Y.

Exercise 3.3 Express "continuity at x" in terms of bases at x and $f(x)$.

Notation The default topology for $\mathbb{R}$ is the standard topology. When we wish to use the lower limit topology, we will indicate the space by $\mathbb{R}_L$.

Exercise 3.4

1. Are the identity maps $\mathbb{R} \to \mathbb{R}_L$ and $\mathbb{R}_L \to \mathbb{R}$ continuous?
2. Consider $f : \mathbb{R} \to \mathbb{R}_L$, $f(x) = x^2$. At which points is it continuous?
3. Can you think of any non-constant continuous map $f : \mathbb{R} \to \mathbb{R}_L$?
4. Consider the step function $f : \mathbb{R}_L \to \mathbb{R}_L$ defined by

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

   Is it continuous?

Exercise 3.5 Let $f : \mathbb{N} \to X$. When can f extend to a continuous function on $\mathbb{N}_\infty$? What if $X = \mathbb{N}$?

Definition 3.6 A function $f : X \to Y$ is called a *homeomorphism* if it is a bijection and both f, f^{-1} are continuous. If such an f exists we say that the spaces X and Y are homeomorphic, and we denote this by $X \simeq Y$.

Exercise 3.7 Each non-empty open interval (a, b) is homeomorphic to $\mathbb{R}$. What about intervals of other types, such as $(a, b]$?

4 Closed Sets

Fix a topological space X.

Definition 4.1 We say $x \in X$ is an *accumulation point* of a subset $F \subset X$ if every neighbourhood U of x contains a point $y \in F$ such that $y \neq x$. The set of accumulation points of F is called its *derived set* and is denoted F'. If $F' \subset F$ we say that F is *closed*.

Theorem 4.2 *$F \subset X$ is closed if and only if F^c is open in X.* □

Definition 4.3 The *closure* $\bar{F}$ of a subset $F \subset X$ is the smallest closed set containing F.

Theorem 4.4 *For any $F \subset X$, $\bar{F} = F \cup F'$.* □

Exercise 4.5 Consider the subset $A = \{(x, \sin(1/x)) : x > 0\}$ of $\mathbb{R}^2$. What is $\bar{A}$?

In a metric space, $x \in \bar{F}$ if and only if there is a sequence (x_n) in F such that $\lim x_n = x$. This is false for general topological spaces:

Exercise 4.6 Consider $\mathbb{R}_{\text{coc}}$, the real numbers with the cocountable topology. Show that a sequence in $\mathbb{R}_{\text{coc}}$ can converge only if it is eventually constant. On the other hand, $F = \mathbb{Q}^c$ satisfies $\bar{F} = \mathbb{R}$.

A related observation is that sequences can no longer capture continuity. If $\lim x_n = x$ implies $\lim f(x_n) = f(x)$, we cannot conclude that f is continuous:

Example 4.7 Consider the identity function $f : \mathbb{R}_{\text{coc}} \to \mathbb{R}$. □

Exercise 4.8 $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed whenever F is closed.

Exercise 4.9 $f : X \to Y$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for every subset A of X.

A set is closed if its closure does not go beyond it. The other extreme is when the closure fills up the whole space.

Definition 4.10 $A \subset X$ is *dense* if $\bar{A} = X$.

Exercise 4.11 A is dense in X if and only if $A \cap U \neq \emptyset$ for each non-empty open $U \subset X$.

5 Subspace Topology

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Is there a natural way to make A into a topological space? One obvious desire is that the inclusion map $\imath : A \to X$, $\imath(x) = x$, be continuous. So we take the coarsest topology on A such that $\imath$ is continuous:

Definition 5.1 The *subspace topology* on A is defined by

$$\mathcal{T}|_A = \{O \cap A : O \in \mathcal{T}\}.$$

Members of $\mathcal{T}|_A$ are said to be *open in* A. The subspace topology is also called the *relative* or *induced topology.* Once A has been equipped with the subspace topology it is called a *subspace* of X.

Exercise 5.2 Check that $\mathcal{T}|_A$ is a topology on A, and that it is the coarsest topology on A such that the inclusion map $\imath$ is continuous.

Exercise 5.3 Let A be a subspace of X. Show that F is closed in A if and only if $F = A \cap C$ where C is closed in X.

Exercise 5.4 Let $S \subset A \subset X$. If S is open in A, must it be open in X? If S is closed in A, will it be closed in X?

Exercise 5.5 Will any assumptions about A guarantee positive answers to the questions in the previous exercise?

Exercise 5.6 Let X be a topological space and $\mathcal{B}$ a basis for its topology. Produce a basis for the subspace topology of $A \subset X$.

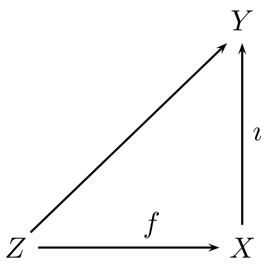
Exercise 5.7 Let X be a topological space, A a subspace of X, and $\mathcal{B}$ a basis at $x \in X$. If $x \in A$, produce a basis at x for the subspace topology of A.

Exercise 5.8 Describe the subspace topology of the given subset of X:

1. $X = \mathbb{R}$, $A = \mathbb{Z}$
2. $X = \mathbb{R}$, $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$
3. $X = \mathbb{R}^2$, $A = \{(x, y) : xy = 0\}$
4. $X = \mathbb{R}^2_L$, $\triangle = \{(x, x) : x \in \mathbb{R}\}$
5. $X = \mathbb{R}^2_L$, $\triangle' = \{(x, -x) : x \in \mathbb{R}\}$

The space $\mathbb{R}^2_L$ denotes the set $\mathbb{R}^2$ with the topology generated by the rectangles of the form $[a, b) \times [c, d)$.

Exercise 5.9 Let $X \subset Y$, with the subspace topology, and $\imath : X \to Y$ be the inclusion map. Then $f : Z \to X$ is continuous if and only if $\imath \circ f : Z \to Y$ is continuous.



This is called the *universal property* of the subspace topology.

Example 5.10 The *n-dimensional sphere* is the following subset of $\mathbb{R}^{n+1}$, with the subspace topology:

$$S^n = \left\{ (x_i)_{i=1}^{n+1} : \sum_i x_i^2 = 1 \right\}$$

□

Example 5.11 Consider the matrix algebra $M(n, \mathbb{R})$. This can be identified with $\mathbb{R}^{n^2}$ and is given a topology via this identification. Its various special subsets, such as $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, and $O(n, \mathbb{R})$, then inherit subspace topologies. The same thing is done for $M(n, \mathbb{C})$. □

Exercise 5.12

1. $GL(n, \mathbb{R})$ is open and dense in $M(n, \mathbb{R})$.
2. $SL(n, \mathbb{R})$, $O(n)$ and $SO(n)$ are closed in $M(n, \mathbb{R})$.
3. $SO(2)$ is homeomorphic to S^1.
4. $SU(2)$ is homeomorphic to S^3.

6 Product Topology

Let X, Y be topological spaces. What is the natural requirement for a topology on $X \times Y$?

Well, we have the projections $\pi_X : X \times Y \to X$, $(x, y) \mapsto x$, and $\pi_Y : X \times Y \to Y$, $(x, y) \mapsto y$, and we would like them to be continuous. For that to happen, we need $X \times V$ and $U \times Y$ to be open in $X \times Y$ whenever U is open in X and V is open in Y. And if this need is satisfied, then it follows that $U \times V$ must be open in $X \times Y$.

Note that the sets $U \times V$, U and V are open, form a basis.

Definition 6.1 The *product topology* on $X \times Y$ is generated by the basis

$$\{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

The discussion before the definition shows that this is the coarsest topology which makes π_X and π_Y continuous.

Exercise 6.2 Is it true that every open set in $X \times Y$ is of the form $U \times V$, where U and V are open?

Next we tackle arbitrary products of topological spaces. Let X_α be topological spaces, with α varying over an index set A. Let $X = \prod_\alpha X_\alpha$ be their Cartesian product. If we copy the definition above, then we might be tempted to use the topology generated by the sets of the form $\prod_\alpha U_\alpha$ where each U_α is open in X_α. This topology would make each projection $\pi_\alpha : X \to X_\alpha$ continuous, however it would not (in general) be the coarsest topology with this property! That distinction belongs to the topology adopted below:

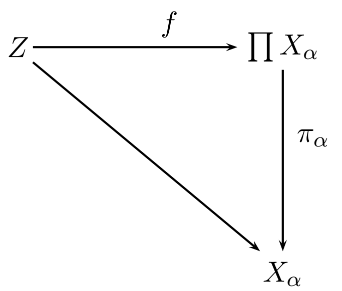
Definition 6.3 The *product topology* on $X = \prod_\alpha X_\alpha$ is generated by the basis

$$\left\{ \prod_\alpha U_\alpha : \begin{array}{l} \text{Each } U_\alpha \text{ is open in } X_\alpha, \\ U_\alpha = X_\alpha \text{ except for finitely many values of } \alpha \end{array} \right\}$$

Exercise 6.4 Verify that the product topology as defined above is the coarsest topology such that each π_α is continuous.

The advantage of using the coarsest topology which makes the projections continuous is that we then get the following **universal property**:

Exercise 6.5 Show that $f : Z \to \prod X_\alpha$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for each α.



Example 6.6 Here is a special case which shows one reason why the product topology is important. The set of all functions $f : X \to Y$ is identified with the product space

$$Y^X = \prod_{x \in X} Y_x\,, \quad Y_x = Y \;\forall x$$

If we give Y^X the product topology, then $f_n \to f$ if and only if $f_n(x) \to f(x)$ for each $x \in X$. Thus we have obtained the *topology of pointwise convergence.* (Note that only Y needs a topology here.) □

Definition 6.7 A function $f : X \to Y$ is *open* it it takes open sets to open sets. It is *closed* if it takes closed sets to closed sets.

Exercise 6.8 The projection map $\pi_Y : X \times Y \to Y$ is open.

Exercise 6.9 The projection map $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto y$, is not closed.

7 Countability Axioms

Consider the following metric space fact:

Let X be a metric space, $x \in X$ and $F \subset X$. Then $x \in \bar{F}$ if and only if there is a sequence (x_n) in F such that $\lim x_n = x$.

Also:

Let $f : X \to Y$ be a function between two metric spaces. If $\lim x_n = x$ implies $\lim f(x_n) = f(x)$, then f is continuous.

Which property of metric spaces is essentially responsible for these facts? By looking at their proofs we see it is because every point has a countable sequence of open balls shrinking down to it. The countability allows the construction of inductive proofs.

Definition 7.1 A topological space X is *first countable* (fc) if there is a countable basis at each point of X.

Exercise 7.2 Prove the following:

1. Metric spaces are fc.
2. $\mathbb{R}_L$ is fc.
3. $\mathbb{R}_{\text{coc}}$ is not fc.

Exercise 7.3 Let there be a countable basis at a point $x \in X$. Then there is a countable basis $\{B_1, B_2, B_3, \dots\}$ at x such that $B_1 \supset B_2 \supset B_3 \cdots$.

Exercise 7.4 Let X be a fc space.

1. Let $F \subset X$. Then $x \in \bar{F}$ if and only if there is a sequence (x_n) in F such that $\lim x_n = x$.
2. Let $f : X \to Y$. Then f is continuous if and only if $\lim x_n = x$ implies $\lim f(x_n) = f(x)$.
3. $A \subset X$ is fc with the subspace topology.
4. If Y is also fc, so is $X \times Y$. (What about an arbitrary product of fc spaces?)

Definition 7.5 A topological space is *second countable* (sc) if it has a countable basis.

Exercise 7.6 A topology generated by countably many subsets is sc.

Exercise 7.7 Let X be a sc space.

1. X is fc.
2. $A \subset X$ is sc with the subspace topology.
3. If Y is also sc, so is $X \times Y$. (What about an arbitrary product of sc spaces?)

Example 7.8 1. $\mathbb{R}$ is sc.

2. An uncountable discrete space is not sc. (So fc $\not\Rightarrow$ sc)
3. $\mathbb{R}_L$ is not sc. (Hint: Consider the subset $\triangle'$ of $\mathbb{R}_L^2$) □

Exercise 7.9 If X is sc then every open cover of X has a countable subcover: If $\mathcal{B}$ is a collection of open sets whose union equals X, then there is a countable subcollection whose union is also X.

Exercise 7.10 If X is sc then every base for the topology of X has a countable subcollection which is also a base.

Suppose we want to establish some property for every point of a space. If the space is countable, we can try to do this by induction. If not, we can only use induction to reach a countable subset A. For other points, we can try to establish the property by approximating them by points from A and using the presence of the property at these points. Of course, this will only have a chance of working if A is dense.

Definition 7.11 X is *separable* if it has a countable dense subset.

Example 7.12

1. $\mathbb{R}$ is separable.
2. $\mathbb{R}_L$ is separable.
3. $\mathbb{R}_{\text{coc}}$ is not separable.
4. Every sc space is separable.

□

Exercise 7.13 A metric space is separable if and only if it is sc.

Exercise 7.14 The topology of $\mathbb{R}_L$ cannot arise from a metric, even though it is Hausdorff and fc. (Hint: Consider $\mathbb{R}_L^2$)

Let us consider the relations that exist between the two countability axioms and separability. In metric spaces, the connection is strong, but in general the only ones we have are that sc implies the other two.

Example 7.15

1. $\mathbb{R}_L$ is separable and fc but not sc.
2. $\mathbb{R}_{\text{cof}}$, the real numbers with cofinite topology, is separable but not fc.
3. $\mathbb{R}_{\text{dis}}$, the real numbers with discrete topology, is fc but not separable.

8 Connectedness

Fix a topological space X. We want to capture the idea of two subsets having a separation, a gap, between them. The Hausdorff property arises out of such considerations – we call two points separated if we can find disjoint neighbourhoods for them. We can generalize this and consider $A, B \subset X$ to be separated from each other if there are disjoint open sets U, V with $A \subset U$ and $B \subset V$. While this is a useful notion, for the moment we adopt a less strict definition:

Definition 8.1 $S \subset X$ is *disconnected* if it can be written as $A \cup B$ such that A, B are non-empty and $A \cap \bar{B} = \bar{A} \cap B = \emptyset$. We call A, B a *disconnection* of S.

Definition 8.2 A set is *connected* if it is not disconnected.

Exercise 8.3 Suppose $S \subset X$ and U, V are disjoint open subsets of X such that $S \subset U \cup V$ and $S \cap U$, $S \cap V$ are non-empty. Show S is disconnected.

Exercise 8.4 Suppose $S \subset X$ is disconnected. Will there necessarily be U, V as in the above exercise?

Exercise 8.5 X is connected if there is no non-empty, proper, subset which is both open and closed.

Exercise 8.6 $S \subset X$ is connected in the subspace topology if and only if it is connected as a subset of X.

Exercise 8.7 X is connected if and only every continuous functon $f : X \to \{\pm 1\}$ is constant. ($\{\pm 1\}$ has the discrete topology)

This characterization is often the most convenient way of showing connectedness. We will emphasize it.

Exercise 8.8

1. $\mathbb{R}$ is connected. (Hint: Intermediate Value Theorem)
2. The connected subsets of $\mathbb{R}$ are the intervals.
3. $GL(n, \mathbb{R})$ and $O(n)$ are disconnected.

Exercise 8.9

1. If A is a connected subset of X and $A \subset B \subset \bar{A}$, then B is connected.
2. If A, B are connected subsets of X and $A \cap B \neq \emptyset$ then $A \cup B$ is connected.
3. Can we weaken the hypothesis of the previous exercise to: A, B are connected and $\bar{A} \cap B \neq \emptyset$?
4. If A_α are connected subsets of X and $A_\alpha \cap A_\beta \neq \emptyset$ for all α, β then $\cup_\alpha A_\alpha$ is connected.
5. $f : X \to Y$ is continuous and X is connected implies that $f(X)$ is connected.

Exercise 8.10 Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. If f is continuous, show that its graph is a connected subset of $\mathbb{R}^2$. Does the converse hold?

Is $\mathbb{R}^2$ connected? Geometric intuition should suggest that it is. It is also easy to prove this by using the results of the last exercise. For instance, $\mathbb{R}^2$ is the union of all the lines passing through origin. These lines are each homeomorphic to $\mathbb{R}$ and hence connected. They are all mutually intersecting, and so by part 4 of the above exercise, $\mathbb{R}^2$ is connected. A variation on this is to consider the x-axis and all the lines perpendicular to it.

Exercise 8.11 If X, Y are connected, so is $X \times Y$.

By induction, we see all finite products of connected spaces are connected.

Exercise 8.12 Consider a product space $X = \prod_\alpha X_\alpha$, where each X_α is connected. Fix $x = (x_\alpha) \in X$.

1. Consider all sets of the form $U_{\beta_1,\ldots,\beta_n} = \prod_\alpha V_\alpha$ where

$$V_\alpha = \begin{cases} X_{\beta_i} & \alpha = \beta_i \\ \{x_\alpha\} & \text{else} \end{cases}$$

   Show each of these sets is connected.

2. $A = \cup U_{\beta_1,\ldots,\beta_n}$ is connected.

3. A is dense in X. Hence X is connected!

Since connectedness is preserved by continuous maps, it provides a useful tool for checking if two given spaces can be homeomorphic.

Exercise 8.13

1. Each S^n, $n \geq 1$, is connected.

2. S^1 is not homeomorphic to $\mathbb{R}$.

3. None of $[0,1]$, $[0,1)$ and $(0,1)$ are homeomorphic to each other.

4. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$. (In fact, $\mathbb{R}^k$ can be homeomorphic to $\mathbb{R}^l$ only if $k = l$. This result is a bit too deep to be reached by our present tools.)

9 Compactness

To motivate the definitions of this section we start with two commonly encountered situations.

- Let $f : X \to \mathbb{R}$ be continuous and bounded. We want to know if it achieves a maximum value. Let

$$M = \sup_{x \in X} f(x)$$

  Then there is a sequence (x_n) in X such that $\lim f(x_n) = M$. If this has a convergent subsequence (x_{n_k}), then $x = \lim x_{n_k}$ must satisfy $f(x) = M$.

- Let X be Hausdorff, $A \subset X$, and $x \notin A$. We want to separate x from A. For each $a \in A$, we can find open and disjoint U_a, V_a such that $x \in U_a$ and $a \in V_a$. Obvious candidates for separating sets are $\cap U_a$ and $\cup V_a$, but the former need not be open. On the other hand, if we can somehow find finitely many $a_i \in A$ such that $A \subset \cup V_{a_i}$, then $\cap U_{a_i}$ and $\cup V_{a_i}$ will do the job.

These two situations lead to two definitions:

Definition 9.1 $A \subset X$ is *sequentially compact* if every sequence in A has a convergent subsequence.

Definition 9.2 $A \subset X$ is *compact* if every open cover of A has a finite subcover: If (U_α) are open and $A \subset \cup U_\alpha$ then $\exists \alpha_1, \ldots, \alpha_n$ such that $A \subset \cup_{i=1}^n U_{\alpha_i}$.

At first sight, these notions may not appear to have much to do with each other. But among the first results one learns in analysis are the Heine-Borel Thorem (the interval $[0,1]$ is compact) and the Bolzano-Weierstrass Theorem (the interval $[0,1]$ is sequentially compact). Both results, moreover, have strikingly similar proofs, proceeding by an infinite sequence of bisections of $[0,1]$. This suggests a close relationship, and indeed we have:

Theorem 9.3 *A metric space X is compact if and only if it is sequentially compact.*

However, this does not hold in general topological spaces.[3] Assuming this fact, which definition should we work with? Since sequences are not good at capturing topological features, we may be biased in favour of compactness over sequential compactness, and indeed this is the more fruitful approach.

Can we have any simple characterization of compact subsets? In $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded. In a general metric space, compact sets are closed and bounded, but are not characterized by this property.

Example 9.4 If X has the discrete metric, then every subset is closed and bounded but only finite subsets are compact. □

Even this partial connection does not hold for general topological spaces.

Example 9.5 Consider $\mathbb{R}_{\text{cof}}$. Then every subset is compact, but only finite subsets are closed. □

[3] We give examples for this at the end of this section.

Things are not completely hopeless:

Exercise 9.6 If X is Hausdorff, then every compact subset of X is closed.

Exercise 9.7 If X is compact, then every closed subset of X is compact.

Exercise 9.8 Show that if A, B are compact subsets of a Hausdorff space X, then $A \cap B$ is compact. Find a counterexample when X is not Hausdorff.

A result which is often used to check for compactness:

Exercise 9.9 X is compact if and only if every family $\{F_\alpha\}$ of closed subsets of X having the finite intersection property satisfies $\cap_\alpha F_\alpha \neq \emptyset$.[4]

Exercise 9.10 If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

Exercise 9.11 If X is compact and $f : X \to \mathbb{R}$ is continuous, then f achieves minimum and maximum values.

Exercise 9.12 Suppose X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection. Then f is a homeomorphism.

Exercise 9.13 (Tube Lemma) Let X be compact. Let $U \subset X \times Y$ contain a slice $X \times \{y_0\}$. Then U contains a *tube* $X \times V$, where V is a neighbourhood of y_0.

Exercise 9.14 Let X be compact. For every Y, the projection $X \times Y \to Y$, $(x, y) \mapsto y$, is a closed map.

Exercise 9.15 Let X, Y be compact. Then the product space $X \times Y$ is compact.

This has a generalization to arbitrary product spaces:

Theorem 9.16 (Tychonoff's Theorem) [5] *Let $\{X_\alpha\}$ be a collection of compact spaces. Then the product space $\prod_\alpha X_\alpha$ is also compact.*

Proof. This is the first proof in these notes which can truly be called non-trivial. It combines the finite intersection property (or FIP) characterization with Zorn's Lemma. We begin with two preliminary lemmas, whose proofs are left to the reader.

---

[4] A family of subsets of X is said to have the finite intersection property if the intersection of finitely many of them is always non-empty.

[5] Other common spellings of Tychonoff are Tychonov and Tikhonov.

Lemma 9.17 *Let X be any set and $\mathcal{B} \subset \wp(X)$ have FIP. Then there is a maximal collection $\mathcal{C} \subset \wp(X)$ such that $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C}$ has FIP.*

Lemma 9.18 *Let $\mathcal{C} \subset \wp(X)$ be a maximal collection having FIP. Then*

1. *$\mathcal{C}$ is closed under finite intersections.*
2. *Let $A \subset X$ such that $A \cap C \neq \emptyset$ for each $C \in \mathcal{C}$. Then $A \in \mathcal{C}$.*

Now let $X = \prod X_\alpha$ and let $\mathcal{B} \subset \wp(X)$ have FIP and consist of closed sets. We have to show that $\cap_{B \in \mathcal{B}} B \neq \emptyset$. By the first lemma, there is a maximal collection $\mathcal{C} \subset \wp(X)$ such that $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C}$ has FIP.

Let $\mathcal{C}_\alpha = \{\overline{\pi_\alpha(C)} : C \in \mathcal{C}\}$. It is easily checked that $\mathcal{C}_\alpha$ has FIP. Since it consists of closed sets, and X_α is compact, it has non-empty intersection:

$$\exists x_\alpha \in \bigcap_{A \in \mathcal{C}_\alpha} A$$

Having chosen such an x_α for each α, we let $x = (x_\alpha) \in X$. We will show this x is common to all elements of $\mathcal{B}$.

Fix $C \in \mathcal{C}$. Choose $y = (y_\alpha) \in C$.

Let O be a subbasic neighbourhood of x, of the form $O = \prod O_\alpha$, where each O_α is open in X_α, and there is an index value β such that $O_\alpha = X_\alpha$ when $\alpha \neq \beta$.

Now $x_\beta \in \overline{\pi_\beta(C)}$ implies $\pi_\beta(C) \cap O_\beta \neq \emptyset$. Let $t \in \pi_\beta(C) \cap O_\beta$.

Define $z = (z_\alpha) \in X$ by

$$z_\alpha = \begin{cases} y_\alpha & \alpha \neq \beta \\ t & \alpha = \beta \end{cases}$$

Then $z_\alpha \in O_\alpha$ for each α, hence $z \in O$. Similarly $z \in C$. Therefore $O \cap C \neq \emptyset$ for each $C \in \mathcal{C}$. By the second lemma, $O \in \mathcal{C}$. Since $\mathcal{C}$ is closed under finite intersections, it also follows that every basic open neighbourhood of x is in $\mathcal{C}$. In turn this establishes that $x \in \bar{C}$ for each $C \in \mathcal{C}$. Since elements of $\mathcal{B}$ are closed and are in $\mathcal{C}$, we have $x \in B$ for each $B \in \mathcal{B}$. □

Example 9.19 For any function $f : [0,1] \to [0,1]$, let $S(f) = \{t : f(t) \neq 0\}$. Consider the space

$$X = \{f : [0,1] \to [0,1] : S(f) \text{ is countable}\}$$

with the topology of pointwise convergence (i.e. consider X as a subspace of $[0,1]^{[0,1]}$ with the product topology). Let (f_n) be a sequence in X. Then $S =$

$\bigcup_n S(f_n)$ is countable. By diagonalization, we see that (f_n) has a subsequence which converges pointwise on S, and hence on all of $[0,1]$.

On the other hand, X is not compact. For each $t \in [0,1]$, define

$$A_t = \{f \in X : f(t) = 1\}$$

The sets A_t are closed and have the finite intersection property, but $\cap_t A_t = \emptyset$. □

Example 9.20 Consider the space $X = \{0,1\}^{\wp(\mathbb{N})}$. As a product of compact spaces, X is compact. Now consider the sequence (f_n) in X defined by

$$f_n(A) = \chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & \text{else} \end{cases}$$

If a subsequence $f_{\alpha(n)}$ converges, then for each $A \subset \mathbb{N}$ the $f_{\alpha(n)}(A)$ are eventually constant. Thus the $\alpha(n)$ are eventually all in A or all outside A. Now choose

$$A = \{\alpha(2), \alpha(4), \alpha(6), \dots\}$$

□

Compactness is useful in separating sets.

Exercise 9.21 Let X be a Hausdorff space, C a compact subset, and $x \in X \setminus C$. Then there exist disjoint open sets U, V such that $x \in U$ and $C \subset V$.

Exercise 9.22 Let X be a Hausdorff space, with disjoint compact subsets C and D. Then there exist disjoint open sets U, V such that $C \subset U$ and $D \subset V$.

10 Locally Compact Spaces

Consider $\mathbb{R}$. This space is not compact, but it can be seen as being made of compact pieces, and this is very useful in its analysis. Of course, every space is made of compact pieces (points!). What is crucial here is that the pieces are large: they can contain open subsets.

Definition 10.1 Let $x \in X$. A *compact neighbourhood* of x is a compact subset of X containing an open neighbourhood of x.

Definition 10.2 A space X is *locally compact* if every point has a compact neighbourhood.

Exercise 10.3 Which of the following spaces are locally compact: $\mathbb{R}$, $\mathbb{R}_{\text{cof}}$, $\mathbb{Q}$?

One can visualize a locally compact space as made of a collection of expanding compact sets, and so be led to a notion of "going to infinity".

Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. We define a new set

$$X_\infty = X \cup \{\infty\}$$

and give it a topology $\mathcal{T}_\infty$ as follows:

$$\mathcal{T}_\infty = \mathcal{T} \cup \{K^c \cup \{\infty\} : K \text{ is a compact subset of } X\}$$

Exercise 10.4 Let X and X_∞ be as above.

1. X_∞ is compact and Hausdorff.
2. The inclusion map $\imath : X \to X_\infty$ is continuous and open.

Therefore X_∞ is called the *one-point compactification* of X.

Exercise 10.5 If X is non-compact, it is dense in X_∞. What happens when X is compact?

Exercise 10.6 The one-point compactification is unique. Let Y be another compact Hausdorff space containing X as a subspace, with $Y \backslash X$ a singleton. Then there is a homeomorphism $\varphi : X_\infty \to Y$ which restricts to the identity map of X.

Example 10.7 We earlier introduced the space $\mathbb{N}_\infty$ and we can now recognize it as the one-point compactification of $\mathbb{N}$. □

Exercise 10.8 The one-point compactification of $\mathbb{R}$ is S^1. In general, $\mathbb{R}^n_\infty \cong S^n$.

Locally compact Hausdorff spaces also allow a simple description of shrinking to a point.

Exercise 10.9 Let X be locally compact and Hausdorff. Let $x \in X$ and V be an open neighbourhood of x. Then there is an open neighbourhood U of x such that $\bar{U}$ is compact and contained in V. (Hint: Use the one-point compactification)

Exercise 10.10 Let X be locally compact and Hausdorff. It has a basis whose elements have compact closures.

We have a version of the Baire Category Theorem (which you may have encountered in the context of complete metric spaces):

Theorem 10.11 (Baire Category Theorem) *Let X be locally compact Hausdorff, and (O_n) a sequence of dense open subsets of X. Then $\cap_n O_n \neq \emptyset$.*

Proof. Pick $x_1 \in O_1$. There is an open neighbourhood U_1 of x_1 such that $\bar{U}_1$ is compact and contained in O_1. Since O_2 is dense, we now pick an $x_2 \in U_1 \cap O_2$, and obtain an open neighbourhood U_2 of x_2 such that $\bar{U}_2$ is compact and contained in $U_1 \cap O_2$. Proceeding in this way, we obtain a decreasing sequence $\bar{U}_n$ of non-empty compact sets. So $\cap_n O_n \supseteq \cap_n \bar{U}_n \neq \emptyset$.

□

Exercise 10.12 Strengthen the conclusion of the last theorem to: $\cap_n O_n$ is dense in X.

Exercise 10.13 Let X be locally compact Hausdorff, and (F_n) a sequence of closed subsets of X such that $X = \cup_n F_n$. Then at least one F_n has non-empty interior.

11 Quotient Topology

A common construction in mathematics is to start with an equivalence relation $\sim$ on a set X, and then consider the *quotient set* $X/\sim$ consisting of the equivalence classes $[x]$ of $\sim$. If X has a structure of a kind, we want to know if the quotient set inherits, in a natural way, the same kind of structure. Thus we are led to quotient groups, quotient rings, quotient vector spaces, etc. Now we want to do the same thing in the context of topological spaces.

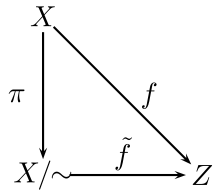
So suppose X is a topological space and $\sim$ an equivalence relation on it. We have the projection map $\pi : X \to X/\sim$, $x \mapsto [x]$. Naturally we want π to be continuous.

Definition 11.1 The *quotient topology* on $X/\sim$ is the finest topology which makes $\pi : X \to X/\sim$ continuous. We call $X/\sim$ a *quotient space* of X.

Exercise 11.2 $V \subset X/\sim$ is open in the quotient topology if and only if $\pi^{-1}(V)$ is open in X.

The quotient topology is also characterized by a universal property:

Exercise 11.3 Suppose $f : X \to Z$ and $\tilde{f} : X/\sim \to Z$ are functions such that $f \circ \pi = \tilde{f}$. Then f is continuous if and only if $\tilde{f}$ is continuous.



Exercise 11.4 Consider a quotient space $X/\sim$.

1. If X is compact, so is $X/\sim$.
2. If X is connected, so is $X/\sim$.

A typical way in which an equivalence relation arises is the following. Start with a function $f : X \to Y$ where X, Y are sets. The equivalence relation on X, *induced by* f, is defined by $x \sim_f x'$ if $f(x) = f(x')$. The map f now induces an injective map $\tilde{f} : X/\sim_f \to Y$. If X and Y are topological spaces, and f is continuous, then the universal property tells us that $\tilde{f}$ is also continuous. If we are lucky, this $\tilde{f}$ may even give a homeomorphism between $X/\sim_f$ and $f(X)$. This construction captures the natural topology for a variety of spaces.

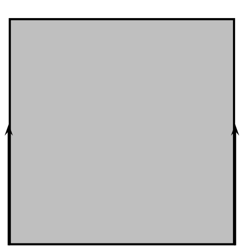
Exercise 11.5 Suppose X, Y are compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous surjective map. Then the quotient space $X/\sim_f$ is homeomorphic to Y.

Note that the key to the above is not really the compactness of X, but the compactness of $X/\sim_f$.

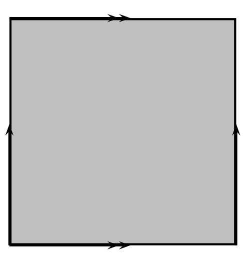
Example 11.6 Consider the map $\exp : \mathbb{R} \to S^1 \subset \mathbb{C}$ defined by $\exp(x) = e^{ix}$. Let $\sim$ be the equivalence relation on $\mathbb{R}$ induced by the map exp. The projection $\pi : X \to X/\sim$, restricted to $[0, 2\pi]$, is continuous and surjective, hence $\mathbb{R}/\sim$ is compact. Therefore $\mathbb{R}/\sim$ is homeomorphic to S^1.

This example shows how to formalize the idea that a circle can be obtained by winding $\mathbb{R}$. A slight variation is to start with $[0, 2\pi]$ – this corresponds to obtaining a circle by sticking together the end points of an interval.

We can similarly obtain a cylinder $S^1 \times \mathbb{R}$ as a quotient of the plane $\mathbb{R}^2$ or of a square I by winding one axis, or a torus $S^1 \times S^1$ by winding both. We represent this by pictures:



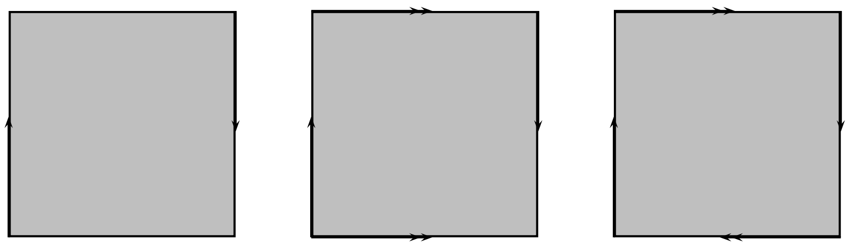
Cylinder



Torus

□

More interesting examples arise when our quotienting introduces twists. For instance we have the spaces represented by the following pictures:



Mobius Strip Klein Bottle Projective Plane

Exercise 11.7 Find maps $f : [0,1] \times [0,1] \to X$ which will induce quotient structures corresponding to the pictures above.

12 Nets

We have noted in various places that outside the context of metric spaces, sequences lose their special place. This is a pity because in that context they do greatly aid in simplifying thought. *Nets* generalize sequences and play the same role in a general topological space.

Definition 12.1 A *directed set* is a set I with a partial order $\leq$ such that for every $i, j \in I$, there is a $k \in I$ with $i, j \leq k$.

Definition 12.2 A *net* in X is a function $f : I \to X$ where I is a directed set. We also denote it as the tuple $(x_i)_{i \in I}$, or $(x_i)_I$, or (x_i), where $x_i = f(i)$.

Example 12.3 Sequences are nets. □

Example 12.4 Let X be a topological space and $\mathcal{B}$ a basis at $x \in X$. Then containment makes $\mathcal{B}$ a directed set: For $U, V \in \mathcal{B}$ we say $U \leq V$ if $U \supset V$. If we now pick one element x_U from each $U \in \mathcal{B}$, we have obtained a net. □

Consider a net $(x_i)_I$ in a topological space X. We say the net *converges* to x, $\lim x_i = x$, if for every open neighbourhood U of x, there is $i \in I$ such that $i \leq j$ implies $x_j \in U$.

Exercise 12.5 Consider the net of Example 12.4. Show that it converges to x.

Exercise 12.6 Let $A \subset X$. Then $x \in \bar{A}$ if and only if there is a net $(x_i)_I$ in A with $\lim x_i = x$.

Exercise 12.7 A function $f : X \to Y$ is continuous if and only if for every convergent net $(x_i)_I$ in X, we have

$$\lim f(x_i) = f(\lim x_i)$$

Definition 12.8 Let I be a directed set and $f : I \to X$ a net. Suppose J is another directed set, with an order preserving map $g : J \to I$, such that $g(J)$ is *cofinal* in I: For every $i \in I$ there is a $j \in J$ with $i \leq g(j)$. Then the net $f \circ g$ is called a *subnet* of f.

Exercise 12.9 If a net in X converges to x, so does every subnet.

Definition 12.10 An element x is a *cluster point* of a net $f : I \to X$ if for every open neighbourhood O of x and $i \in I$, $\exists j \in I$ such that $j \geq i$ and $f(j) \in O$.

Exercise 12.11 A net $f : I \to X$ has x as a cluster point if and only if it has a subnet which converges to x.

(HINT: If x is a cluster point, consider $M = \{(i, O) : i \in I,\ O \text{ is an open neighbourhood of } x,\ i \in O\}$.)

Theorem 12.12 *X is compact if and only if every net in X has a convergent subnet.*

Proof. We use the finite intersection property (FIP) characterization of compactness. First, suppose every net has a convergent subnet. Let $\mathcal{C}$ be a collection of closed subsets of X which has FIP. Order $\mathcal{C}$ by containment: $A < B$ if $B \subset A$. For each $C \in \mathcal{C}$ choose $x_C \in C$. Then $f : \mathcal{C} \to X$, $f(C) = x_C$, is a net. It has a convergent subnet, i.e. $\exists$ directed set $\mathcal{D}$ and an order preserving map $g : \mathcal{D} \to \mathcal{C}$ such that $g(J)$ is cofinal, and the net $f \circ g$ converges to $x \in X$. We leave it to you to check that x belongs to each member of $\mathcal{C}$.

Next, suppose X is compact and $f : \mathcal{C} \to X$ is a net. For each $C \in \mathcal{C}$ we define a subset of X:

$$F_C = \{f(D) : D \geq C\}$$

The collection $\{F_C\}$ has FIP. Hence the collection $\{\bar{F}_C\}$ has FIP. So its intersection $\bigcap_C \bar{F}_C$ has an element x.

The element x is easily seen to be a cluster point of f. Hence there is a subnet converging to it. □

Exercise 12.13 Suppose X is compact. Then every net has a convergent subnet. So every sequence has a convergent subsequence. So X is sequentially compact. Evaluate this argument.

13 Topological Groups

Definition 13.1 A *topological group* is a group G with a topology such that the group operations (multiplication and inverse)

$$\begin{aligned} m &: G \times G \to G,\ (x, y) \mapsto xy \\ \imath &: G \to G,\ x \mapsto x^{-1} \end{aligned}$$

are continuous.

This connection between algebra and topology is fruitful for the study of both.

Example 13.2 Some topological groups ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$):

1. $(\mathbb{K}, +)$ with the standard topology.
2. $(\mathbb{K}^*, \cdot)$ with the standard topology.
3. $GL(n, \mathbb{K})$ with matrix multiplication and the subspace topology from $M(n, \mathbb{K}) \cong \mathbb{K}^{n^2}$. □

From now on, we will just call our first two examples $\mathbb{K}$ and $\mathbb{K}^*$.

Exercise 13.3 Are $\mathbb{R}_L$, $\mathbb{R}_{\text{cof}}$ and $\mathbb{R}_{\text{coc}}$ topological groups when addition is taken as the group operation?

Exercise 13.4 Consider a group G with a topology. Show it is a topological group if and only if the map

$$G \times G \to G,\ (x, y) \mapsto xy^{-1}$$

is continuous.

Every element $a \in G$ leads to a *left multiplication* map $l_a : G \to G$ defined by $x \mapsto ax$. It also leads to a *right multiplication* map $r_a : G \to G$ defined by $x \mapsto xa$.

Exercise 13.5 Each multiplication map (l_a or r_a) is a homeomorphism. Hence, so is conjugation: $x \mapsto axa^{-1}$.

Definition 13.6 Let G be a topological group and H a subgroup. If we equip H with the subspace topology from G, then H becomes a topological group in its own right, and we call it a *topological subgroup* of G.

Example 13.7 $SL(n,\mathbb{R})$, $O(n)$ and $SO(n)$ are topological subgroups of the general linear group $GL(n,\mathbb{R})$. $GL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $U(n)$ and $SU(n)$ are topological subgroups of $GL(n,\mathbb{C})$. □

Exercise 13.8 Let G be a topological group and H a subgroup.

1. $\bar{H}$ is a subgroup of G.
2. If H is a normal subgroup, so is $\bar{H}$.
3. If H is open, it is also closed.
4. If H is closed and has finite index, then it is also open.

Exercise 13.9 What are the open subgroups of $\mathbb{R}$? Of $\mathbb{Q}$?

Since translations are homeomorphisms, they allow us to translate information at one point to other points. In particular, we focus on what happens around the identity element e. Let $\mathcal{U}$ be the set of all the neighbourhoods of identity.

Exercise 13.10 The open sets of G are of the form xU, $x \in G$, $U \in \mathcal{U}$. If $\mathcal{B}$ is any base at e, then $x\mathcal{B}$ is a base at x.

Exercise 13.11 For every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $VV \subset U$ and $V = V^{-1}$. (Subsets satisfying $A = A^{-1}$ are called *symmetric*)

Exercise 13.12 G is Hausdorff if and only if $\cap_{U \in \mathcal{U}} U = \{e\}$.

Exercise 13.13 Let G, H be topological groups and $\pi : G \to H$ a group homomorphism. Then π is continuous if and only if it is continuous at e.

Definition 13.14 A map $\pi : G \to H$ between topological groups is a homomorphism if it is continuous and a group homomorphism.

Let G be a topological group and H a topological subgroup. Let $\pi : G \to G/H$ be the canonical projection, and equip the coset space G/H with the quotient topology: $V \subset G/H$ is open if and only if $\pi^{-1}(V)$ is open.

Exercise 13.15 Let H be a topological subgroup of G.

1. The canonical projection $\pi : G \to G/H$ is open.

2. If H is a normal subgroup of G, then G/H with the quotient topology is a topological group.

Exercise 13.16 We now look at how some topological properties behave on passage to the coset space. We already know compactness and connectedness are preserved.

1. Let H be a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed. (Note that it doesn't matter whether G is Hausdorff!)
2. If H and G/H are connected, so is G.
3. If H and G/H are compact, what about G?

Exercise 13.17 Let $G = SL(2,\mathbb{R})$, $K = SO(2)$, $A = \{\begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} : t > 0\}$. Show the multiplication map $K \times A \to G$ is a homeomorphism, and hence $SL(2,\mathbb{R})$ is connected.[6]

Exercise 13.18 The group $GL^+(2,\mathbb{R})$ of matrices with positive determinant, is connected. Hence $GL(2,\mathbb{R})$ has two connected components.

Exercise 13.19 Similarly show $SL(2,\mathbb{C})$ and $GL(2,\mathbb{C})$ are connected.

From group theory, we know the importance of group actions. In fact groups are important because they act on other objects, and by their action reveal the structure of these objects.

Definition 13.20 Let G be a topological group and X a topological space. A *continuous action* of G on X is a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, such that

1. $g \cdot (h \cdot x) = (gh) \cdot x$, $\forall g, h \in G$, $\forall x \in X$.
2. $e \cdot x = x$, $\forall x \in X$.

From now on we will take the continuity for granted, and just use the term *action*. Also we will write gx for $g \cdot x$.

For a while we forget about topologies and consider some pure algebra.

Definition 13.21 Let G act on X. Then

[6]This way of factoring the members of $SL(2,\mathbb{R})$ is called the *Iwasawa decomposition*. It generalizes to a much bigger class of groups, called *reductive groups*, which includes $GL(n,\mathbb{R})$ and every connected subgroup of $GL(n,\mathbb{R})$ which is closed under transpose.

1. The action is *transitive* if $\forall x, y \in X$ there is a $g \in G$ such that $gx = y$.
2. The *orbit* of $x \in X$ is the set $\mathcal{O}_x = \{gx : g \in G\}$.
3. The *stabilizer* or *isotropy subgroup* of $x \in X$ is $G_x = \{g \in G : gx = x\}$.

Example 13.22 Let H be a subgroup of G. It acts on G by right multiplication: $h \cdot g = gh^{-1}$. The orbits are the right cosets of H. □

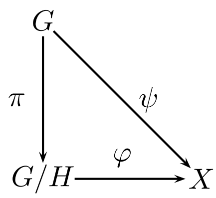
Example 13.23 Let H be a subgroup of G. Then G acts on G/H by left multiplication: $g \cdot (xH) = (gx)H$. This action is transitive, and the isotropy subgroup of eH is H. □

Exercise 13.24 Let G act on X. Then

1. Define a relation on X by $x \sim y$ if there is a $g \in G$ such that $gx = y$. Show this is an equivalence relation and its equivalence classes are the orbits $\mathcal{O}_x$.
2. Suppose the action is transitive. Fix $x \in X$ and let H be its isotropy subgroup. Then there is a bijection $\varphi : G/H \to X$ given by $gH \mapsto gx$.

Now bring back the topology: Let a topological group G act on a topological space X. We can ask questions about the quotient topology of $X/\sim$. Or, when the action is transitive, a natural question is whether the bijection of X with G/H is a homeomorphism.

Consider the bijection $\varphi : G/H \to X$ when the action is transitive and H is an isotropy subgroup. We have a commuting diagram



where $\psi(g) = gx$. Since the action is continuous, so is ψ, and hence also φ. So for establishing homeomorphism one has to show that φ is open.

Exercise 13.25 Let a compact group G act transitively on a Hausdorff space X. Let H be the isotropy group of $x \in X$. Then G/H is homeomorphic to X.

Example 13.26 Consider the orthogonal group $SO(n)$, acting on $\mathbb{R}^n$ by matrix multiplication (we view members of $\mathbb{R}^n$ as column vectors). This restricts to a transitive action on the unit circle S^{n-1}. Now consider the "north pole" $N = (0, \dots, 0, 1)$. Its isotropy subgroup H consists of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \qquad A \in SO(n-1)$$

Thus, $H \cong SO(n-1)$. So we have a homeomorphism

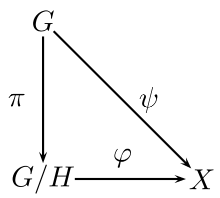
$$SO(n)/SO(n-1) \cong S^{n-1}$$

Since $SO(1)$ is connected, we can show from this that each $SO(n)$ is connected! □

Exercise 13.27 Show each unitary group $U(n)$ is connected.

Theorem 13.28 *Let G be a locally compact, Hausdorff and second countable topological group, acting transitively on a locally compact and Hausdorff space X. Let H be the isotropy subgroup of a fixed $x \in X$. Then G/H and X are homeomorphic.*

Proof. Consider the commuting diagram



$$\pi(g) = gH$$
$$\psi(g) = g \cdot x$$
$$\varphi(gH) = g \cdot x$$

We have to show that the map φ is open (we already know it is a continuous bijection). It suffices to show ψ is open. Let g be a member of an open set $U \subset G$. We have to show $\psi(g)$ is in the interior of $\psi(U)$. Since the action by g^{-1} is a homeomorphism, this is equivalent to showing that x is in the interior of $g^{-1}U \cdot x$, and $g^{-1}U$ is an open neighbourhood of e.

So we have to show that if V is an open neighbourhood of e, then x is in the interior of $V \cdot x$.

Since G is locally compact and Hausdorff, there is an open neighbourhood O of e such that $K = \bar{O}$ is compact, $K = K^{-1}$, and $K^2 \subset V$. Since G is sc, there exist countably many $g_1, g_2, \dots \in G$ such that $g_i O$ cover G. Hence $G = \cup_i g_i K$. It follows that

1. $X = \bigcup_i g_i K \cdot x$. (Transitivity of the action)
2. Each $g_i K \cdot x = \psi(l_{g_i}(K))$ is compact, hence closed. (X Hausdorff)

By the Baire Category Theorem, one of the $g_i K \cdot x$ has non-empty interior. Let $k \in K$ such that $g_i k \cdot x$ is in the interior of $g_i K \cdot x$. But then x is in the interior of $k^{-1}K \cdot x \subset K^2 \cdot x \subset V \cdot x$. □

Notation

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}^*$	The number field $\mathbb{K}$ with zero removed
$\subset$	Contained in (allows equality)
$\subsetneq$	Strictly contained in (forces inequality)
$\wp(X)$	Power set of X
$M(n, \mathbb{K})$	The $n \times n$ matrices, over $\mathbb{K}$
$GL(n, \mathbb{K})$	General linear group, the $n \times n$ invertible matrices over $\mathbb{K}$
$SL(n, \mathbb{K})$	Special linear group, the $n \times n$ matrices over $\mathbb{K}$ with determinant one
$O(n)$	Orthogonal group, the $n \times n$ real orthogonal matrices
$SO(n)$	Special orthogonal group, the orthogonal matrices with determinant one
$U(n)$	Unitary group, the $n \times n$ complex unitary matrices
$SU(n)$	Special unitary group, the unitary matrices with determinant one

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