Representations of Real Reductive Groups - IX

Amber Habib

Dec 22, 1997

1 Introduction

We have described in the last lecture how knowledge of the discrete series is crucial to wider classifications. Therefore we would like to have a construction of the discrete series. Since this series is seen as the generalisation of the compact case, let us start by writing down a theorem which constructs the irreducible representations of a compact group K.

Let T be a Cartan subgroup of K, and $\mathfrak{b}$ a Borel subalgebra containing $\mathfrak{t}$. Then $\mathfrak{b}$ gives a K-invariant complex structure on K/T. Now let (τ, W) be an irreducible representation of T. Extend its derivative to $\mathfrak{b}^-$ by letting $\mathfrak{n}^-$ act trivially. Let $\mathcal{W}$ be the resulting holomorphic line bundle on K/T, and $V = \Gamma(K/T, \mathcal{W})$ the space of holomorphic sections. K acts on V by left translation π.

Theorem 1.1 (Borel-Weil) *(π, V) is non-zero iff (τ, W) is a dominant representation of T. The non-zero (π, V) exhaust the irreducible representations of K.* □

An alternative construction is as follows. Let τ be a dominant regular metaplectic representation of $\tilde{T}$. Give K/T the complex structure from $\mathfrak{b}^-$, with (μ, W) the representation $\tau \otimes \rho$ of T. Letting $\mathfrak{n}$ act by 0, this becomes a $(\mathfrak{b}, T)$-module, and gives a holomorphic vector bundle $\mathcal{W}$ on G/T. It happens that $\Gamma(K/T, \mathcal{W}) = 0$, but we can also consider the K action on the higher cohomology groups.

Write $\mathcal{O}_{\mathcal{W}}$ for the sheaf of germs of holomorphic sections of $\mathcal{W}$. The cohomology group $H^p(K/T, \mathcal{W})$ is defined to be the pth sheaf cohomology group of K/T with coefficients in $\mathcal{O}_{\mathcal{W}}$. It is isomorphic to the cohomology defined using the $\bar{\delta}$ operator on $(0, p)$ forms with coefficients in $\mathcal{W}$.

Theorem 1.2 (Bott-Borel-Weil) *Let $s = \dim_{\mathbb{C}} K/T$. Then $H^s(K/T, \mathcal{W}) \cong$ irreducible representation of highest weight $\tau \otimes (-\rho)$.* □

Generalising the first construction to non-compact reductive G yields representations that contain finite-dimensional summands — hence they are rarely unitary, and never square-integrable. On the other hand, the cohomology groups are harder to analyse for non-compact G as they are usually infinite-dimensional. Nevertheless, Schmid(1967 & 1975) showed that the $H^s(G/T,\mathcal{W})$ yield the discrete series.

A philosophical problem with the last result is that the proof uses Harish-Chandra's classification of discrete series, rather than constructing them *a priori*. Kostant and Langlands conjectured that this could be overcome by replacing Dolbeault cohomology by a kind of space of L^2 harmonic forms. That discrete series can be so realised was also proven by Schmid(1976). This construction has not been extended to other representations, except for a few special cases, due to the analytic difficulties.

This section was condensed from Chapter 5 of Vogan[5].

Notation. In the rest of this lecture we will use the subscript $_0$ to denote real Lie algebras, and drop the subscript to denote their complexifications. Thus a real Lie group G will have Lie algebra $\mathfrak{g}_0$, whose complexification will be $\mathfrak{g}$.

2 Zuckerman Functor

Zuckerman proposed and carried out much of a programme in which the problems of the analytic approach are side-stepped by working in the category $\mathcal{C}(\mathfrak{g},K)$ of $(\mathfrak{g},K)$-modules. A useful and allied category is $\mathcal{C}(\mathfrak{g},\mathfrak{k})$ — it consists of $\mathfrak{g}$-modules in which all the vectors are $\mathfrak{k}$-finite. If K is connected, a $(\mathfrak{g},K)$-module is just a $(\mathfrak{g},\mathfrak{k})$-module with K-integrable weights. If K is also simply connected, then the $(\mathfrak{g},K)$-modules are just the $(\mathfrak{g},\mathfrak{k})$-modules.

Let $\mathfrak{q}$ be a parabolic subalgebra in $\mathfrak{g}$, with Levi decomposition $\mathfrak{l}+\mathfrak{u}$, and corresponding Levi subgroup L in G. If L is θ-stable, then either it is contained in the Levi factor of a proper real parabolic subgroup of G, or $\mathfrak{q}$ is θ-stable. The first case corresponds to parabolic induction as we have defined it so far. Now we consider the second.

So fix a parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ of $\mathfrak{g}$, with each of $\mathfrak{q}$, $\mathfrak{l}$, $\mathfrak{u}$ being θ-stable. $\mathfrak{l}_0$ is a real form of $\mathfrak{l}$, with corresponding Levi subgroup L in G. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition given by θ. Then $\mathfrak{u}=\mathfrak{u}\cap\mathfrak{k}+\mathfrak{u}\cap\mathfrak{p}$. Let $r=\dim(\mathfrak{u}\cap\mathfrak{p})$ and $s=\dim(\mathfrak{u}\cap\mathfrak{k})$.

The problem now is to construct a $(\mathfrak{g},K)$-module starting from a $(\mathfrak{l},L\cap K)$-module V. First we make V into a $(\mathfrak{q},L\cap K)$-module by letting $\mathfrak{u}$ act trivially. Now consider the *produced module*

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(V)=\mathrm{Hom}_{\mathfrak{q}}(\mathcal{U}(\mathfrak{g}),V)$$

It is a $\mathfrak{g}$-module under the action $(X\cdot\phi)(u)=\phi(uX)$, $X\in\mathfrak{g}$. $L\cap K$ also acts on the produced module, by $(b\cdot\phi)(u)=b\cdot(\phi(Ad(b^{-1}u))$. This action is not $L\cap K$-finite, but we get a $(\mathfrak{g},L\cap K)$-module by considering the subspace of $L\cap K$-finite vectors. We sum this up by writing

$$X=\mathrm{pro}_{(\mathfrak{q},L\cap K)}^{(\mathfrak{g},L\cap K)}(V)=\mathrm{Hom}_{\mathfrak{q}}(\mathcal{U}(\mathfrak{g}),V)_{L\cap K}$$

The final step requires going from the $(\mathfrak{g}, L\cap K)$-module X to a $(\mathfrak{g}, K)$-module. If K and $L\cap K$ are connected, and K is also simply connected, then we need to go from a $(\mathfrak{g}, \mathfrak{l}\cap\mathfrak{k})$-module to a $(\mathfrak{g}, \mathfrak{k})$-module and this is easily achieved by just picking out the $\mathfrak{k}$-finite vectors in X.

In general, we have to go through a few more hoops. The subspace $\tilde{\Gamma}_o(X)$ of $\mathfrak{k}$-finite vectors in X is acted on by the universal cover $\tilde{K_o}$ of the identity component of K. Let Z be the kernel of the covering map. Then K_o has an action π on

$$\Gamma_o(X) = \{v \in \tilde{\Gamma}_o(X) \ : \ z\cdot v = v \ \forall z \in Z\}$$

Let ρ be the $L\cap K$-action on X. Define

$$\Gamma_1(X) = \{v \in \Gamma_o(X) \ : \ \pi(m)v = \rho(m)v \ \forall m \in L\cap K_o\}$$

Let K_1 be the subgroup of K generated by K_o and $L\cap K$. Then it has a unique action on $\Gamma_1(X)$ extending π and ρ. Finally, set

$$\Gamma(X) = \mathrm{Ind}_{K_1}^{K}(\Gamma_1(X))$$

The action of $\mathfrak{g}$ is given by

$$(X\cdot f)(k) = (Ad(k^{-1})(X))\cdot(f(k))$$

and this makes $\Gamma(X)$ into a $(\mathfrak{g}, K)$-module. Γ is the *Zuckerman functor* from $\mathcal{C}(\mathfrak{g}, L\cap K)$ to $\mathcal{C}(\mathfrak{g}, K)$. Putting Γ and pro together, we get a left exact functor

$$\mathcal{R}_{\mathfrak{q},L}(X) = \Gamma^{(\mathfrak{g},K)}_{(\mathfrak{g},L\cap K)} \circ \mathrm{pro}^{(\mathfrak{g},L\cap K)}_{(\mathfrak{q},L\cap K)}(V)$$

from $\mathcal{C}(\mathfrak{l}, L\cap K)$ to $\mathcal{C}(\mathfrak{g}, K)$. Due to all the subspaces involved in the definition of $\Gamma(X)$, it is quite likely to be the zero space. As in the analytic picture, we can get around this by going to cohomology.

3 Derived Zuckerman Functors

$\mathcal{R}$ is covariant and left exact, hence it will have right derived functors if $\mathcal{C}(\mathfrak{l}, L\cap K)$ has enough injectives; i.e. every member has an injective resolution.

The proof of this for $\mathcal{C}(\mathfrak{l}, \mathfrak{l}\cap\mathfrak{k})$ can be found in Enright and Wallach[1]. Consider any $W\in\mathcal{C}(\mathfrak{l}\cap\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k})$, and let $I(W) = \mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(\mathcal{U}(\mathfrak{l}), W)_{\mathfrak{l}\cap\mathfrak{k}}$. For $f\in I(W)$, define $\iota(f) = f(1)$. Then ι induces an $\mathfrak{l}\cap\mathfrak{k}$-module projection from $I(W)$ onto W. In turn, this gives a natural bijection

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{l})}(V, I(W)) \to \mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(V, W) \text{ for } V\in\mathcal{C}(\mathfrak{l}, \mathfrak{l}\cap\mathfrak{k})$$

This shows that $I(W)$ is injective in $\mathcal{C}(\mathfrak{l}, \mathfrak{l}\cap\mathfrak{k})$. Further, if $V\in\mathcal{C}(\mathfrak{l}, \mathfrak{l}\cap\mathfrak{k})$ and $v\in V$, then define $f_v\in I(V)$ by $f_v(x) = x\cdot v$. The map $v\mapsto f_v$ is an injective $\mathfrak{l}$-module map, $V\hookrightarrow I(V)$. Hence $\mathcal{C}(\mathfrak{l}, \mathfrak{l}\cap\mathfrak{k})$ has enough injectives and the right derived functors $\mathcal{R}^i$ exist. The same argument shows that $\mathcal{C}(\mathfrak{g}, \mathfrak{l}\cap\mathfrak{k})$ has enough injectives, while the functor pro is exact, and so we also have

$$\mathcal{R}^i \cong \Gamma^i \circ \mathrm{pro}$$

A useful fact is that the derived Zuckerman functors are well behaved with respect to restriction to $\mathfrak{k}$. Namely, let Γ denote the Zuckerman functors from $\mathcal{C}(\mathfrak{g}, \mathfrak{l}\cap\mathfrak{k})$ to $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ as well as from $\mathcal{C}(\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k})$ to $\mathcal{C}(\mathfrak{k}, \mathfrak{k})$. Let $\mathcal{F}$ be the forgetful functors from $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ to $\mathcal{C}(\mathfrak{k}, \mathfrak{k})$ and from $\mathcal{C}(\mathfrak{g}, \mathfrak{l}\cap\mathfrak{k})$ to $\mathcal{C}(\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k})$. Then the following holds for $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{l}\cap\mathfrak{k})$:

$$\mathcal{F} \circ \Gamma^i(V) \cong \Gamma^i \circ \mathcal{F}(V)$$

Now the derived functors $\Gamma^i : \mathcal{C}(\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k}) \to \mathcal{C}(\mathfrak{k}, \mathfrak{k})$ are given by a relatively simple resolution. The *Koszul resolution* of the trivial $\mathfrak{k}$-module is[1]

$$\begin{aligned} 0 \to \mathbb{C} \quad &\to \quad \mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(\mathcal{U}(\mathfrak{k}), X_0) \to \cdots \\ &\to \quad \mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(\mathcal{U}(\mathfrak{k}), X_{2s}) \to 0 \end{aligned}$$

where $X_n = \Lambda^n(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^*)$. Let $V \in \mathcal{C}(\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k})$. Tensoring by V and using the natural isomorphism $\mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(\mathcal{U}(\mathfrak{k}), W) \otimes V \cong \mathrm{Hom}_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})}(\mathcal{U}(\mathfrak{k}), W \otimes V)$, we get an injective resolution of V. Applying Γ, and dropping the first term gives the complex from which the Γ^i are calculated:

$$0 \to \mathbb{C} \to I(X_0 \otimes V) \to \cdots \to I(X_{2s} \otimes V) \to 0$$

Theorem 3.1 *For $V \in \mathcal{C}(\mathfrak{k}, \mathfrak{l}\cap\mathfrak{k})$ and $i > 2s$, $\Gamma^i(V) = 0$.* □

The lifting of these constructions and results to the $(\mathfrak{g}, K)$ level is done through the use of an associative algebra which essentially does for $(\mathfrak{g}, K)$-modules what $\mathcal{U}(\mathfrak{g})$ does for $\mathfrak{g}$-modules. For motivation, consider the category $\mathcal{C}(\mathfrak{k}, K)$. Let C_K be the algebra of all linear combinations of matrix coefficients of finite-dimensional representations of K, with convolution as multiplication. In general C_K lacks an identity, but it has an "approximate identity" in the following manner: For each irreducible representation τ of K, let $\chi_\tau = (\deg \tau)(\text{character of } \tau)$. For each finite set F of equivalence classes of equivalence classes of irreducible representations of K, set $\chi_F = \sum_{\tau\in F} \chi_\tau$. Now for any $c \in C_K$, and sufficiently large F, $c\chi_F = c = c * \chi_F$ (Take F to be large enough so it contains all the irreducibles contributing to c). A left C_K-module V is called *unital* if for each $x \in V$ and large enough F, $\chi_F x = x$. And then $C_K x$ is finite dimensional for all $x \in V$.

Lemma 3.2 *Every $(\mathfrak{k}, K)$-module is a unital left C_K-module in a natural way, and conversely. For any two such V and W, $\mathrm{Hom}_{(\mathfrak{k},K)}(V, W) = \mathrm{Hom}_{C_K}(V, W)$.* □

[1] We skip all the important information such as the formulae for the actions and the differentials.

Consider the set

$$R(\mathfrak{g}, K) = C_K \otimes_{\mathfrak{k}} \mathcal{U}(\mathfrak{g})$$

$R(\mathfrak{g}, K)$ has a C_K action on the left and a $\mathcal{U}(g)$ action on the right. It is made into a complex associative algebra by the multiplication

$$(c \otimes u)(c' \otimes u') = c * (c'(\cdot)Ad(\cdot)^{-1}u) \otimes u'$$

As usual we do not prove that all this is well-defined, or the next important result:

Theorem 3.3 *Every $(\mathfrak{g}, K)$-module is a unital left $R(\mathfrak{g}, K)$-module in a natural way, and conversely. For any two such V and W,* $\mathrm{Hom}_{(\mathfrak{g},K)}(V, W) = \mathrm{Hom}_{R(\mathfrak{g},K)}(V, W)$. □

We note that the approximate identity in $R(\mathfrak{g}, K)$ consists of the $e_F = \chi_F \otimes 1$, where the χ_F are as before. The lifting from the $(\mathfrak{g}, \mathfrak{k})$ level to the $(\mathfrak{g}, K)$ level is now done essentially by replacing $\mathcal{U}(g)$ by $R(\mathfrak{g}, K)$ in all the constructions and proofs.

4 Bernstein Functor and Duality

Suppose we have an inclusion of pairs $(\mathfrak{q}, H) \subset (\mathfrak{g}, K)$. Then $R(\mathfrak{q}, H)$ is not naturally a subalgebra of $R(\mathfrak{g}, K)$, but $R(\mathfrak{g}, K)$ is an $R(\mathfrak{q}, H)$-module (from left and right). For $V \in \mathcal{C}(\mathfrak{q}, H)$, we define

$$\begin{aligned} P(V) &= P_{(\mathfrak{q},H)}^{(\mathfrak{g},K)}(V) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{q},H)} V \\ I(V) &= I_{(\mathfrak{q},H)}^{(\mathfrak{g},K)}(V) = \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g}, K), V) \end{aligned}$$

P and I are covariant functors from $\mathcal{C}(\mathfrak{q}, H)$ to $\mathcal{C}(\mathfrak{g}, K)$. P is right exact and I is left exact. The functors pro and Γ are special cases of I. The corresponding special cases for P are given below:

$$\begin{aligned} \mathrm{ind}(V) &= \mathrm{ind}_{(\mathfrak{q},L\cap K)}^{(\mathfrak{g},L\cap K)}(V) = P_{(\mathfrak{q},L\cap K)}^{(\mathfrak{g},L\cap K)}(V) \\ \Pi(W) &= P_{(\mathfrak{g},L\cap K)}^{(\mathfrak{g},K)}(W) \end{aligned}$$

The derived functors are given by a projective resolution. The analogues of the $\mathcal{R}^i$ are $\mathcal{L} = \Pi \circ \mathrm{ind}$ and its derived functors $\mathcal{L}_i \cong \Pi_i \circ \mathrm{ind}$. Π is called the *Bernstein functor*. The following duality theorem is crucial:

Theorem 4.1 *Let $0 \leq i \leq 2s$. Then there is a natural isomorphism of functors:*

$$\Pi_i \cong \Gamma^{2s-i}$$

□

In general, the derived functors need not give representations that are irreducible or unitary or even non-zero. Viewing them in terms of Γ^i is suitable for investigating irreducibility, while the the Π_i are useful for unitarity questions.

5 Vanishing Theorems

Recall that $\Gamma^i = 0$, and hence $\mathcal{R}^i = 0$, if $i > 2s$ ($s = \dim_{\mathbb{C}} \mathfrak{u} \cap \mathfrak{k}$). In fact one can narrow down the list of possible nonzero $\mathcal{R}^i$ considerably further. A first result is that $\mathcal{R}^i = 0$ if $i > s$ — this is also proven by working at the $(\mathfrak{k}, K)$ level. We can even have $\mathcal{R}^i(Z) = 0$ for every $i < s$. This will happen if Z satisfies certain positivity conditions:

Theorem 5.1 *There exists a constant* $C = C(\mathfrak{q})$ *such that* $\mathcal{R}^i(Z) = 0$ *for* $i < s$ *whenever all* $L \cap K$ *types* τ_μ *of* Z *satisfy*

$$\langle \mu, \alpha \rangle \geq C \ \forall \alpha \in \triangle(\mathfrak{u} \cap \mathfrak{k})$$

□

The ideal theorem then is one where one has vanishing both above and below s, while $\mathcal{R}^s(Z)$ is nonzero and irreducible unitarisable. An example of this is the following (from Knapp and Vogan[3]):

Let $Z \in \mathcal{C}(\mathfrak{l}, L \cap K)$ be one-dimensional with weight λ. Define $Z^\# = Z \otimes \Lambda^{\mathrm{top}} \mathfrak{u}$ and $V = \mathrm{ind}_{(\mathfrak{q}^-, L \cap K)}^{(\mathfrak{g}, L \cap K)} Z^\#$.

Theorem 5.2 *Suppose* $\mathrm{Re}\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle \geq 0$ *for all* $\alpha \in \triangle(\mathfrak{u})$. *Suppose also that* Z *is unitarisable. Then*
1. $\Pi_i(V) \cong \Gamma^i(V) = 0$ *for* $i \neq s$.
2. $\Pi_s(V) \cong \Gamma^s(V)$ *is unitarisable.* □

Note that this theorem says nothing about irreducibility. But if Z satisfies the stronger condition that $\mathrm{Re}\langle \lambda + \rho, \alpha \rangle \geq 0$ for all $\alpha \in \triangle(\mathfrak{u})$, then we can also claim that $\Gamma^s(V)$ is irreducible when Z is.

References

[1] T. J. Enright and N. R. Wallach. Notes on homological algebra and representations of lie algebras. *Duke Mathematical Journal*, 47(1):1–15, 1980.

[2] A. W. Knapp. *Lie Groups, Lie Algebras, and Cohomology*, volume 34 of *Mathematical Notes*. Princeton University Press, Princeton, New Jersey, 1988.

[3] A. W. Knapp and D. Vogan. *Cohomological Induction and Unitary Representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1995.

[4] D. Vogan. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in Mathematics*. Birkhäuser, Boston, 1981.

[5] D. Vogan. *Unitary Representations of Reductive Lie Groups*, volume 118 of *Annals of mathematics Series*. Princeton University Press, Princeton, New Jersey, 1987.