Representations of Real Reductive Groups - VIII

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1 Introduction

Today we take stock and look at possible routes ahead. We have established the main construction of representations – parabolic induction – and done the standard examples, namely $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$. In this lecture, we will describe some of the general appearance and state of representation theory for real reductive groups. We mainly follow the exposition in Collingwood[1], which is a review essay on Wallach[5]. Let us fix a standard minimal parabolic $P = MN$, with $M = {}^0MA$ called a *Levi subgroup* of G.

2 The Subrepresentation Theorem

A fundamental result of Harish-Chandra, the *Subquotient theorem*, asserts that any irreducible representation π of G is infinitesimally equivalent to a subquotient (ie, a quotient of a subspace) of a principal series representation $U(P,\sigma,\lambda)$. This was refined by Casselman to the *Subrepresentation theorem*:

Theorem 2.1 *Every irreducible $(\mathfrak{g}, K)$-module is equivalent to a subrepresentation of some $U(P,\sigma,\lambda)$.* □

Let us recall that induction takes us from admissible representations of reductive subroups M to admissible representations of G. To get a better picture of the situation, we also need the reverse process of reduction: going from admissible representations of G to admissible representations of M. Consider the functor that takes an admissible representation π of G, with underlying Harish-Chandra module H_K, to the finite dimensional space $H_0(\mathfrak{n},\pi) = H_K/(\mathfrak{n}\cdot H_K)$ of $\mathfrak{n}$-coinvariants in π. Let $U(P,\sigma,\lambda)$ have representation space V, induced from $\sigma\otimes e^{\lambda}\otimes 1$ acting on V^{σ}. The two processes are related by *Frobenius reciprocity*:

$$\mathrm{Hom}_{\mathfrak{g},K}(H_K, V_K) = \mathrm{Hom}_{\mathfrak{m},{}^0M}(H_0(\mathfrak{n},\pi), V^{\sigma}_{{}^0M}).$$

The key to the embedding result is the following:

Theorem 2.2 *If π is irreducible then $H_0(\mathfrak{n},\pi)\neq 0$.* □

Viewing $H_0(\mathfrak{n},\pi)$ as a finite-dimensional representation of the abelian group A, there must be A-invariant spaces

$$0 = S_0 \subset S_1 \subset \cdots \subset S_t = H_0(\mathfrak{n},\pi)$$

such that each S_j/S_{j-1} is a one-dimensional character $e^{\nu(\log a)}$, $a \in A$, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. We define the space of *homology exponents of* π:

$$\mathcal{A}_{hom}(\pi) = \{\nu \ : \ e^{\nu(\log a)} \text{ arises as a subquotient } S_j/S_{j-1} \text{ of } H_0(\mathfrak{n},\pi)\}$$

Casselman's approach to proving $\mathcal{A}_{hom}(\pi) \neq 0$ involves analysis of the matrix coefficients of H_K. Write such a matrix coefficient as $f_{v,w}(g) = \langle w, \pi(g)v\rangle$, $g \in G$, $v \in H_K$, and $w \in \hat{H}_K$ [1]. It turns out to have an expansion in exponentials of form $e^{\nu(\log a)}$; so one can define

$$\mathcal{A}_{asy}(\pi) = \{\nu \ : \ e^{\nu(\log a)} \text{ arises in the expansion of some } f_{v,w} \text{ for } \pi\}$$

Casselman found non-zero elements of $\mathcal{A}_{asy}(\pi)$ that were also in $\mathcal{A}_{hom}(\pi)$. These *leading exponents* provide the λ we seek. If ν is such a leading exponent, let $\lambda = \nu + \rho$. This leading exponent also provides a non-zero G map from H into the space of the induced representation

$$\mathrm{Ind}^G_{AN}(e^\lambda \otimes 1) = \mathrm{Ind}^G_{{}^0MAN}[(\mathrm{Ind}^{{}^0M}_1(1)) \otimes e^\lambda \otimes 1]$$

Now $\mathrm{Ind}^{{}^0M}_1(1) = L^2({}^0M)$ decomposes discretely by the Peter-Weyl Theorem. So we get a G map into $U(P,\sigma,\lambda)$ for some σ. The map is injective because H is irreducible.

However, Wallach[5] avoids the use of exponents for proving Theorem 2.2, relying instead on the subquotient theorem (for connected groups of inner type) and some rough asymptotic estimates for the spherical principal series[2].

3 Growth of Matrix Coefficients

As we have just mentioned, the growth of matrix coefficients is crucial to the subrepresentation theorem. Control on the growth is also crucial for the theory of intertwining operators among the principal series representations. One can define formal integral operators which are clearly intertwining, if they exist. Existence, or convergence, demands bounds on the growth of matrix coefficients. The rates of growth also lead to the classification of the irreducible representations, and it is this aspect we will look at now.

There are two key inequalities on the growth, called *weak* and *strong* respectively. As the expressions are messy we do not write them out here but give

[1] $\hat{H}_K$ is the K-finite dual of H_K.

[2] Spherical principal series are induced from the trivial one-dimensional $\sigma \in \widehat{{}^0M}$.

the equivalent algebraic conditions. Let V be a finitely generated admissible $(\mathfrak{g}, K)$-module. Let $\triangle_0 = \{\alpha_1, \ldots, \alpha_r\}$. Define $H_1, \ldots, H_r \in \mathfrak{a}$ by $\alpha_i(H_j) = \delta_{ij}$. Then define $\Lambda_V \in \mathfrak{a}^*$ by

$$\Lambda_V(H_j) = \max\{-\mathrm{Re}\ \mu(H_j) \mid \mu \in \mathcal{A}_{hom}(\hat{V})\}$$

Theorem 3.1 *Consider a Hilbert representation H of G. The following are equivalent:*
1. $\Lambda_{H_K} + \rho \in -\mathrm{Cl}(\mathfrak{a}^{*+})$.
2. H satisfies the weak inequality. □

An irreducible representation of G having these properties is called *tempered.* The terminology comes from the fact that its global character, as defined by Harish-Chandra, is tempered as a distribution.

Theorem 3.2 *Consider a Hilbert representation H of G. The following are equivalent:*
1. $\Lambda_{H_K} + \rho \in -(\mathfrak{a}^{*+})$.
2. H satisfies the strong inequality. □

Irreducible representations of this kind are called *square integrable*, as their matrix coefficients turn out to be square integrable. Hence they are always unitary.

Theorem 3.3 (Harish-Chandra, Langlands, & Trombi) *Consider a tempered representation H of G. Then there exists a standard p-pair (P_F, A_F), an irreducible square integrable representation σ of ${}^0M_F$, and $\mu \in (\mathfrak{a}_F)^*$, such that H is a summand of $U(P_F, \sigma, i\mu)$. Conversely, every such irreducible summand is tempered.* □

Consequently, tempered representations are also unitary. Another characterisation that results from this theorem is that their K-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$.

Tempered representations are also characterised by the property of appearing in the direct integral decomposition of $L^2(G)$. Square integrable representations occur as discrete summands in this decomposition and are therefore called the *discrete series.* They represent the best possible generalisation of the compact situation.

4 Langlands Classification

Consider a standard p-pair (P_F, A_F). Let (σ, H_σ) be a tempered representation of ${}^0M_F$. Let $\mu \in (\mathfrak{a}_F)^*_{\mathbb{C}}$ be such that $\mathrm{Re}(\mu, \alpha) > 0$ for $\alpha \in \triangle(P_F, A_F)$. The triple (P_F, σ, μ) is called *Langlands data.* In this situation, there is a nonzero intertwining operator $j_{F,\sigma,\mu} : U(P_F, \sigma, \mu) \to U(P_F, \sigma, \mu)$ whose image $J_{F,\sigma,\mu}$ is isomorphic to the unique irreducible quotient of $U(P_F, \sigma, \mu)$.

Theorem 4.1 *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then there exists Langlands data (P_F, σ, μ) such that V is $(\mathfrak{g}, K)$-isomorphic with $J_{P_F,\sigma,\mu}$. Further, if V is also isomorphic to $J_{P_{F'},\sigma',\mu'}$ for a Langlands data $(P_{F'}, \sigma', \mu')$, then $F = F'$, $\mu = \mu'$, and σ is unitarily equivalent to σ'* $\square$

Thus the classification of the irreducible $(\mathfrak{g}, K)$-modules reduces first to that of the tempered ones and then, because of Theorem 3.3 and double induction, to that of the discrete series. We also need a list of the summands of the $U(P_F, \sigma, i\mu)$ in Theorem 3.3.

Note that this can be seen as a loosening of the subquotient and subrepresentation theorems, in that the irreducible admissible representations are not realised in principal series, but in more general induced representations.

Example $SL(2, \mathbb{R})$

Recall from Lecture II the partition of the irreducible $(\mathfrak{g}, K)$-modules into the classes $D(\mu, \pm)$, $F(\mu)$, and $P(c; \varepsilon + 2\mathbb{Z})$. In terms of our present classification, the $D(\mu, \pm)$, $\mu \neq 0$, form the discrete series. The mock discrete series $D(0, \pm)$ and the first and second principal series complete the set of tempered representations. Indeed, by Theorem 4.2 of Lecture II, we have $P(c; \varepsilon + 2\mathbb{Z}) \cong U(B, \varepsilon, \sqrt{c})$ for these principal series. In fact that result gives exactly the realisation of all the irreducibles as Langlands quotients. The non-tempered unitary representations, which do not therefore show up in $L^2(G)$, are the complementary series and the trivial representation $F(1)$.

5 Discrete Series and Cohomological Induction

Harish-Chandra classified the discrete series through an analysis of the characters of the tempered representations. Wallach[5] however constructs the discrete series directly, using the method of *cohomological parabolic induction*. This method has its roots in attempts to construct geometric realisations of discrete series through Dolbeault cohomology groups, in the spirit of the Bott-Borel-Weil theorem for compact groups. The striking success of Schmid in constructing such realisations, showed also how difficult it would be to do so for more general representations. In the 70's, Zuckerman proposed an algebraic version of this approach, which after some recasting has taken the form of cohomological parabolic induction, or derived Zuckerman functors. The power of this approach resides in its consisting of two dual families of derived functors. The original ones, due to Zuckerman, are left exact and suitable for settling irreducibility. The others, due to Bernstein, are right exact and suited for unitarity questions. The duality allows us to use whichever ones are convenient.

The classification of discrete series will be the subject of the next lecture, so for now we say no more about this topic.

6 The Unitary Dual

While the tempered representations solve the main problem as far as harmonic analysis is concerned, from the point of view of representation theory the full unitary dual still remains to be determined in general. A survey of the situation till 1981 can be found in Knapp and Speh [3]. This paper also contains a condition for unitarity involving the intertwining operator used in the Langlands classification. But the condition is hard to verify in general.

Before the early 70's, successes in finding $\hat{G}$ involved a few classical groups of small size, as well as $SO_e(n,1)$ and $SU(n,1)$. Advent of the Langlands classification and the derived Zuckerman functors has made possible more general results. Thus the unitary duals of the following groups are now known: $SU(2,2)$ (Knapp, 1981), linear semisimple Lie groups of real rank one with compact Cartan subgroups (Baldoni Silva and Barbasch, 1983), $\overline{SO_e}(2n,2)$ (Angelopoulos, 1983), $GL(n,F)$ for $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ (Vogan, 1986), complex classical Lie groups (Barbasch, 1989), $\widetilde{GL}(n,\mathbb{R})$ (Huang, 1990), and G_2 (Vogan, 1994).

Apart from such case-by-case solutions, attempts have also been made to generate large classes of unitary representations across the different group families. Enright, Howe and Wallach(1983) have classified the irreducible unitary representations of highest weight. These are all obtained either as discrete series or by continuation of the discrete series parameter[3]. The authors use Howe's theory of dual pairs, which relates subclasses of the unitary duals of certain pairs of subgroups of symplectic groups. The exact size and constitution of these subclasses in general is a matter still under investigation.

Among the host of other achievements that deserve mention, one must single out the $\mathcal{D}$-module approach of Beilinson and Bernstein, as it brings together many earlier approaches under one roof. For instance, once the theory is in place, the Langlands classification is "easily" obtained. However, its connections with the problem of deciding unitarity are not yet understood.

References

[1] D. H. Collingwood. What is non-commutative harmonic analysis? *http://www.math.washington.edu/˜colling/Research/Review/bkrv.ps.*

[2] A. W. Knapp. *Representation Theory of Semisimple Groups*, volume 36 of *Princeton Mathematical Series.* Princeton University Press, Princeton, New Jersey, 1986.

[3] Recall from Lecture II that the irreducible unitary highest weight representations of $SL(2,\mathbb{R})$ are the discrete series, the mock discrete series, and the trivial representation.

[3] A. W. Knapp and B. Speh. Status of classification of irreducible unitary representations. In *Harmonic Analysis*, volume 908 of *Lecture Notes in Mathematics*. Springer-Verlag, 1982.

[4] A. W. Knapp and D. Vogan. *Cohomological Induction and Unitary Representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1995.

[5] N. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1988.