Representations of Real Reductive Groups - VII

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1 Introduction

In the previous lecture we had studied parabolic subgroups, after claiming that inducing from the representations of these subgroups yields a large class of irreducible representations of the full group. In this lecture, we justify that claim.

The reader might find it useful to take another look at Lectures II and V. Much of the material here is a straightforward generalisation of that material.

Following the general approach in these lectures, we give constructions in some detail, but skip or greatly abbreviate proofs of theorems.

2 Induction

In this section we describe the general process of constructing induced representations. Let H be a closed subgroup of G and (π, V) a representation of H. We have the associated vector bundle $G \times_H V$ over G/H which is obtained by quotienting $G \times V$ by the equivalence relation $(gh, v) \sim (g, \pi(h)v)$, where $g \in G$, $h \in H$, and $v \in V$. Sections of this bundle can be viewed as functions $f : G \to V$ which satisfy $f(gh) = \pi(h)^{-1} f(g)$, for each $g \in G$, and $h \in H$. G acts on such functions by translation to give a representation of G. If we start with a unitary representation of H and desire this G action to be unitary as well, we need a G-invariant measure on G/H, which does not exist in general. However, let δ be the modular function of H and V_π be the measurable functions $G \to V$ s.t.

$$f(gh) = \delta(h)^{-1/2}\pi(h)^{-1} f(g) \ , h \in H, \text{and } \int_{G/H} ||f(x)||^2 dx < \infty$$

Let G act by translation.[1] Then V_π is a representation of G which is unitary when V is unitary. We write $V_\pi = \mathrm{Ind}_H^G(V)$.

[1] In Lectures II and V we had defined induction slightly differently, the δ contribution coming in the G-action rather than in the definition of V_π. This is a matter of taste alone, and we make the change for consistency with our references from here on.

3 Parabolic Induction

We will start with a representation of a parabolic subgroup $P = MN$ and apply the induction procedure we have just defined. To carry this out, we need to know the modular function δ_P of P. Recall first that since P is a Lie group, we have the formula $\delta_P(p) = |\det Ad_P(p)|$. Consider the Langlands decomposition $P = {}^0MAN$. By definition,

$$\delta_P(m) = |\det Ad_P(m)| = 1$$

for $m \in {}^0M$. Now $\mathfrak{n}$ is nilpotent, so $\exp : \mathfrak{n} \to N$ is surjective, and further each $ad(X)$, $X \in \mathfrak{n}$, is nilpotent on $\mathfrak{p}$. So we have

$$\delta_P(n) = |\det Ad_P(n)| = e^{\mathrm{tr}(ad_P(\log n))} = e^0 = 1$$

Let ρ be half the sum of the restricted roots occurring in $\mathfrak{n}$. For $p = man$, we get

$$\begin{aligned}\delta_P(p) &= \delta_P(a) = |\det Ad_P(a)| = e^{\mathrm{tr}(ad_P(\log a))} \\ &= e^{\mathrm{tr}(ad_N(\log a))} = e^{2\rho(\log a)}\end{aligned}$$

Now let σ be an irreducible unitary representation of ${}^0M$ on V, $\lambda \in (\mathfrak{a}^*)_{\mathbb{C}}$, and $\pi = \sigma \otimes \exp\lambda \otimes 1$ the corresponding representation of P on V. The above calculations give:

$$\begin{aligned}\mathrm{Ind}_P^G(V) = \{f : G \xrightarrow{\text{mble}} V \ : \quad & f(xp) = e^{-(\lambda+\rho)(\log a)}\sigma(m)^{-1}f(x), \\ & \textstyle\int_K ||f(k)||^2 dk < \infty\}\end{aligned}$$

The G action is as before: $(g \cdot f)(x) = f(g^{-1}x)$.

3.1 Compact picture

Now note that $G = KP$ (the decompositions $g = kp$ are not unique, however). This indicates the possibility of framing the induction procedure in terms of certain functions on K. First, we observe that the elements of $\mathrm{Ind}_P^G(V)$ defined above are in bijection with the functions $F \in L^2(K)$ which satisfy the following invariance condition:

$$F(km) = \sigma(m)^{-1}F(k) \ \forall k \in K, \ m \in {}^0M \cap K$$

The bijection is given in one direction by restriction to K. In the other direction, given F as above, we define $f \in \mathrm{Ind}_P^G(V)$ by

$$f(kman) = e^{-(\lambda+\rho)(\log a)}\sigma(m)^{-1}F(k)$$

As for the G action, we get it by first finding the f corresponding to a given F, acting on it, and then restricting to K. Corresponding to $G = K\,{}^0MAN$, let $g = \kappa(g)\mu(g)e^{H(g)}n$. Then we find

$$(g \cdot F)(k) = e^{-(\lambda+\rho)H(g^{-1}k)}\sigma(\mu(g^{-1}k))^{-1}F(\kappa(g^{-1}k))$$

While the action is now more complicated, the advantage is that the representation space is described only in terms of σ.

3.2 Noncompact picture

Recall that $\bar{N}P$ is an open dense submanifold of G, and so we can restrict to $\bar{N}$ without loss. The representation space is $L^2(\bar{N}, e^{2\mathrm{Re}\,\lambda H(\bar{n})}\,d\bar{n})$. Let $g \in \bar{N}MAN$ decompose as $g = \bar{n}(g)m(g)a(g)n$. Then the action is

$$(g \cdot f)(\bar{n}) = e^{-(\lambda+\rho)\log\, a(g^{-1}\bar{n})}\sigma(m(g^{-1}\bar{n}))^{-1}f(\bar{n}(g^{-1}\bar{n}))$$

The advantage here is that $\bar{N}$ being an euclidean space, analysis is easier.

3.3 Some facts

Let us use $U(P, \sigma, \lambda)$ to denote the representations we have just constructed.

1. $U(P, \sigma, \lambda)$ is a continuous representation of G.

2. If λ is imaginary, then $\sigma \otimes \exp\lambda \otimes 1$ is unitary, and so is $U(P, \sigma, \lambda)$.

3. (Double Induction) Let ${}^0M_*A_*N_*$ be a parabolic subgroup of ${}^0M$, so that ${}^0M_*(A_*A)(N_*N)$ is a parabolic subgroup of G. If σ is a unitary representation of ${}^0M_*$ and λ_*, λ are in $(\mathfrak{a}_*^*)_{\mathbb{C}}$ and $(\mathfrak{a}^*)_{\mathbb{C}}$ respectively, then there is a canonical isomorphism

$$\begin{aligned}&\mathrm{Ind}_{{}^0MAN}^{G}(\mathrm{Ind}_{{}^0M_*A_*N_*}^{{}^0M}(\sigma \otimes \lambda_* \otimes 1) \otimes \exp\lambda \otimes 1)\\ &\quad\cong \mathrm{Ind}_{{}^0M_*(A_*A)(N_*N)}^{G}(\sigma \otimes \exp(\lambda_* + \lambda) \otimes 1)\end{aligned}$$

which preserves norms.

4 Principal Series

When P is a minimal parabolic, the representations $U(P, \sigma, \lambda)$ are said to be in the *principal series*. If λ is imaginary, we say $U(P, \sigma, \lambda)$ is in the *unitary principal series.*[2]

4.1 Intertwining operators

To study the irreducibility of the principal series representations, as well as the isomorphisms that may exist amongst them, we need to look at the intertwining operators. Bruhat used his double-coset decomposition to analyse the intertwining operators among the $U(P, \sigma, \lambda)$ in terms of the intertwining operators among the $\sigma \otimes \lambda \otimes 1$. A first result is:

[2]Some authors use *nonunitary prinicipal series* for the general case, and *principal series* for the special case.

Theorem 4.1.1 *For imaginary λ the dimension of the space of operators intertwining $U(P,\sigma,\lambda)$ with itself is $\leq |W_{\sigma,\lambda}|$, where*

$$W_{\sigma,\lambda} = \{\tilde{w} \in W(G,A) \ : \ w\sigma \cong \sigma, \ w\lambda = \lambda\},$$

the $W(G,A)$ actions being $w\sigma(m) = \sigma(w^{-1}mw)$ and $w\lambda(H) = \lambda(Ad(w)^{-1}H)$. □

Corollary 4.1.2 *Fix σ. Then $U(P,\sigma,\lambda)$ is irreducible for an open dense set of λ (λ imaginary).* □

More generally, Bruhat theory shows that there can be a nonzero intertwining operator between $U(P,\sigma,\lambda)$ and $U(P,\sigma',\lambda')$ only if $\sigma' \cong w\sigma$ and $\lambda' = w\lambda$ for some $w \in N_K(A)$.

Example. For $SL(2,\mathbb{R})$, $W(G,A)$ has order 2, and $w\sigma$ always equals σ, while $w\lambda$ equals $\pm\lambda$. So the only possible isomorphisms are between $\pi_{0,iv}$ and $\pi_{0,-iv}$, and between $\pi_{1,iv}$ and $\pi_{1,-iv}$. From the algebraic classification in Lecture II, we know such isomorphisms exist.

The problem of actually constructing such isomorphisms between $U(P,\sigma,\lambda)$ and $U(P,w\sigma,w\lambda)$ in general requires the use of distributions and the study of the rates of decay of the matrix coefficients of the representation. One defines a formal operator, an integral, which is intertwining whenever it exists. Then it can be shown that this integral converges for large enough $\mathrm{Re}\lambda$. Finally, one can use analytic continuation to extend its definition to all λ. We do not go into more details at present on the pretext that they will be covered later, in more generality, when we look at Harish-Chandra's theory of characters.

4.2 Infinitesimal character

Another way of calculating possible isomorphisms among the principal series representations is to calculate their infinitesimal characters. First, recall that we have

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \oplus (\mathfrak{n}\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) + \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^{-})$$

and then $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \oplus \mathfrak{n}\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.Let p be the projection of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ onto $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$. Define an algebra homomorphism $u : \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ by extending $u(H) = H + \rho(H)$, $H \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$. Then $\gamma = u \circ p : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ gives the Harish-Chandra isomorphism onto $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})^W$.

Now suppose $P = MN_P$ is a standard parabolic subgroup of G, with $M = {}^0MA$. Let $\mathfrak{t}$ be a θ-stable Cartan subalgebra of ${}^0\mathfrak{m}$, and set $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, so that $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

As before, let $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\triangle_0 = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$, with consistent orderings so that $\triangle_0^+ = \triangle^+|_{\mathfrak{a}} - 0$. Then $\triangle_M = \triangle({}^0\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ consists of members of $\triangle$ vanishing on $\mathfrak{a}$, and $\triangle_M^+ = \triangle^+ \cap \triangle_M$. The half-sums of positive roots satisfy $\rho = \rho_0 + \rho_M$.

Since $\mathfrak{m}$ is itself reductive, also with Cartan subalgebra $\mathfrak{h}$, we have corresponding maps $p_m : \mathcal{Z}(\mathfrak{m}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ and $u_m : \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ giving a Harish-Chandra isomorphism $\gamma_m = u_m \circ p_m$. Further, we have

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \bar{\mathfrak{n}} \Longrightarrow \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \subset \mathfrak{n}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \oplus \mathcal{U}(\mathfrak{m}_{\mathbb{C}})$$

Let μ be the projection of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ onto $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$. Then $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ implies $\mu(Z) \in \mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$ and the maps defined so far are related by

$$p = p_m \circ \mu$$

Theorem 4.2.1 *Let $\sigma \in \widehat{{}^0M}$ have infinitesimal character Λ_σ relative to $\mathfrak{t}_{\mathbb{C}}$. Then $U(P, \sigma, \lambda)$ has infinitesimal character $\Lambda_\sigma + \lambda$ relative to $\mathfrak{h}_{\mathbb{C}} = (\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}$.*

Proof. For $g \in G$ and $f \in$ induced space, $g \cdot f$ is right-invariant under N. Hence for $X \in \mathfrak{n}$,

$$X \cdot (g \cdot f)(1) = \frac{d}{dt} g \cdot f((\exp tX)^{-1})\Big|_{t=0} = 0$$

Therefore $\mathfrak{n}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ annihilates f and for $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$,

$$Z \cdot f(1) = \mu(Z) \cdot f(1)$$

Now consider $X = X_a + X_m \in \mathfrak{a} \oplus {}^0\mathfrak{m}$.

$$\begin{aligned} X \cdot f(1) &= \frac{d}{dt}(e^{\lambda+\rho_0} \otimes \sigma)(\exp tX) f(1)\Big|_{t=0} \\ &= ((\lambda + \rho_0)(X_a) + \sigma(X_m)) \cdot f(1) \end{aligned}$$

Now we can calculate the inf. character. Let $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$. then

$$\begin{aligned} Z \cdot f(1) &= \mu(Z) \cdot f(1) = ((\lambda + \rho_0) \otimes \sigma)(\mu(Z)) f(1) \\ &= (\lambda + \rho_0 + \Lambda_\sigma)(\gamma_m(\mu(Z))) f(1) \\ &= (\lambda + \rho_0 + \Lambda_\sigma)(u_m \circ p(Z)) f(1) \\ &= (\lambda + \rho_0 + \Lambda_\sigma + \rho_M)(p(Z)) f(1) \\ &= (\lambda + \Lambda_\sigma + \rho)(p(Z)) f(1) = (\Lambda_\sigma + \lambda)(u \circ p(Z)) f(1) \\ &= (\Lambda_\sigma + \lambda)(\gamma(Z)) f(1) = \chi_{\Lambda_\sigma + \lambda}(Z) f(1) \end{aligned}$$

□

References

[1] A. W. Knapp. *Representation Theory of Semisimple Groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1986.

[2] D. A. Vogan. *Unitary Representatons of Reductive Lie Groups*, volume 118 of *Annals of Mathematical Studies*. Princeton University Press, Princeton, New Jersey, 1987.

[3] N. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1988.