Representations of Real Reductive Groups - VI

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1 Introduction

In earlier talks, we have seen the importance of representations induced from those of subgroups. The principal series for $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ was obtained in this fashion. This construction has its origin in the work of Mackey (in the 1950's) on obtaining representations of a locally compact group G from those of closed subgroups. The "Mackey machine" is at its most powerful when G has a large normal subgroup N. Reductive groups, on the other hand, do not have large normal subgroups. For example, the only non-trivial normal subgroup of $SL(2,\mathbb{R})$ is $\{\pm I\}$. Saviours arrive in the form of the parabolic subgroups, even though they are self-normalising and hence far from normal.

In this lecture, we describe the parabolic subgroups of a reductive group.

2 Some Structure of G

First we recall Wallach's description[3] of a reductive group. Let G_C be a subgroup of $GL(n,\mathbb{C})$ defined as the common zeros of a finite set of complex polynomials on $M_n(\mathbb{C})$, and closed under complex conjugation. Let $G_R = G_C \cap GL(n,\mathbb{R})$. Then G is a real reductive group if it is a finite cover of an open subgroup G_0 of G_R.

On G_R we have the automorphism $\theta(g) = (g^{-1})^*$. Identifying the Lie algebra $\mathfrak{g}$ of G with that of G_R, we see it has the automorphism $\theta(X) = -X^*$. This is called a *Cartan involution*. Let $\mathfrak{k}$ be the $+1$ eigenspace for θ in $\mathfrak{g}$, and $\mathfrak{p}$ the -1 eigenspace. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a *Cartan decomposition* of $\mathfrak{g}$. We have $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$.

We have seen in our examples the importance of the interplay of the actions of the maximal compact subgroup K with Lie algebra $\mathfrak{k}$, and our choice of Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. An important aspect was that for $SL(2,\mathbb{R})$, but not for $SL(2,\mathbb{C})$, we were able to choose $\mathfrak{h} \subset \mathfrak{k}$ s.t. $\mathfrak{h}_{\mathbb{C}}$ was a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. So our first task is to look into the possible relationships of $\mathfrak{h}$ and $\mathfrak{k}$.

Since $\mathfrak{g}$ is a linear Lie algebra, we have the form $B(X,Y) = \mathrm{Tr}XY$. It is easily checked to be $\mathfrak{g}$ and θ invariant, and comparison with the inner product $\mathrm{Tr}XY^*$ shows it is non-degenerate. It behaves just as the Killing form does for the semisimple case.

Definition 2.0.1 *We say $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. A θ-stable Cartan subalgebra $\mathfrak{h}$ is "maximally split" if $\mathfrak{h} \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$, and it is "fundamental" if $\mathfrak{h} \cap \mathfrak{k}$ is maximal abelian in $\mathfrak{k}$.*

All maximally split Cartan subalgebras are conjugate to each other through elements of $Int(\mathfrak{g})$. All fundamental Cartan subalgebras are similarly conjugate to each other.[1]

2.1 Construction of a maximally split Cartan subalgebra

Let $\mathfrak{a}$ be maximal among all the abelian subalgebras of $\mathfrak{g}$ that are contained in $\mathfrak{p}$. $H \in \mathfrak{a}$ implies $H^* = -\theta(H) = H$ and so H is semisimple. Therefore $\mathfrak{a}$ acts semisimply on $\mathfrak{g}$. Let $\mathfrak{m}$ be the centraliser of $\mathfrak{a}$ in $\mathfrak{g}$. It is θ-invariant, and so decomposes into ± 1 eigenspaces: $\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}) \oplus (\mathfrak{m} \cap \mathfrak{p})$. By definition, $\mathfrak{m} \cap \mathfrak{p} = \mathfrak{a}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{m} \cap \mathfrak{k}$. Now the elements of $\mathfrak{t}$ are skew-adjoint, hence they do not act semisimply on $\mathfrak{g}$ — but they act semisimply on $\mathfrak{g}_{\mathbb{C}}$. Therefore $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ acts semisimply on $\mathfrak{g}_{\mathbb{C}}$, and in fact $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It is now obvious that $\mathfrak{h}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$.

2.2 Construction of a fundamental Cartan subalgebra

Let $\mathfrak{t}_1$ be a maximal abelian subalgebra of $\mathfrak{k}$. Let $\mathfrak{a}_1$ be a maximal abelian subspace of $\mathfrak{p}$ subject to $[\mathfrak{t}_1, \mathfrak{a}_1] = 0$. Then $\mathfrak{h}_1 = \mathfrak{a}_1 \oplus \mathfrak{t}_1$ is a fundamental Cartan subalgebra of $\mathfrak{g}$.

2.3 Iwasawa decomposition

Let $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{t}$ be as in Section 2.1, so that $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a maximally split Cartan subalgebra. For $\mu \in \mathfrak{a}^*$ set

$$\mathfrak{g}^{\mu} = \{X \in \mathfrak{g} \ : \ [H, X] = \mu(H)X \ \forall H \in \mathfrak{a}\}$$

The set

$$\triangle_0 = \triangle(\mathfrak{g}, \mathfrak{a}) = \{\mu \in \mathfrak{a}^* \ : \ \mu \neq 0, \ \mathfrak{g}^{\mu} \neq 0\}$$

is called the *restricted root system*. Let $H_0 \in \mathfrak{a}$ be s.t. $\mu(H_0) \neq 0, \ \forall \mu \in \triangle_0$. Set $\triangle_0^+ = \{\mu \in \triangle \ : \ \mu(H_0) > 0\}$. Define $\mathfrak{n} = \sum_{\mu \in \triangle_0^+} \mathfrak{g}^{\mu}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is the *Iwasawa decomposition* of $\mathfrak{g}$. Let $K_0 = \{g \in G_0 \ : \ \theta(g) = g\}$ and K its pre-image in G. Then K is a maximal compact subgroup of G. Let A and N be the connected subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$.

Theorem 2.3.1 *The map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a surjective diffeomorphism.*

[1] Alternative terminology here is "maximally non-compact" and "maximally compact".

2.4 Split component

The centre $\mathfrak{z}$ of $\mathfrak{g}$ is θ-invariant, and so $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}) \oplus (\mathfrak{z} \cap \mathfrak{p})$. Then $Z = \exp(\mathfrak{z} \cap \mathfrak{p})$ is called the *split component* of G.

Let $X(G)$ denote the group of continuous homomorphisms $G \to \mathbb{R}^\times$. Set ${}^0G = \{g \in G \ : \ |x(g)| = 1, \ \forall x \in X(G)\}$.

Theorem 2.4.1 *The map $Z \times {}^0G \to G$, $(z, g) \mapsto zg$, is a Lie group isomorphism.* □

3 Parabolic Subgroups

We stay in the situation of Section 2.3. Let $\triangle = \triangle(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ be the root system of $\mathfrak{g}_\mathbb{C}$ w.r.t. $\mathfrak{h}_\mathbb{C}$. Then

$$\triangle_0 = \triangle|_\mathfrak{a} - \{0\}$$

For $\alpha \in \triangle$, let H_α be the dual element in $\mathfrak{h}_\mathbb{C}$. Define $\mathfrak{h}_\mathbb{R}$ to be the real span of $\{H_\alpha \ : \ \alpha \in \triangle\}$. Consider $H_1 \in \mathfrak{h}_\mathbb{R}$ s.t. $\mu(H_1) \neq 0$ for all $\mu \in \triangle_0$. Extend to a basis $H_1, \ldots, H_r$ of $\mathfrak{h}_\mathbb{R}$. Then we have the corresponding lexicographic ordering on $\triangle$ and a choice of positive roots $\triangle^+$, with base Ψ. On the other hand, H_1 gives a choice of positive roots $\triangle_0^+$ of $\triangle_0$, with base Ψ_0. The choices are related by

$$\begin{aligned} \triangle_0^+ &= \triangle^+|_\mathfrak{a} - \{0\} \\ \Psi_0 &= \Psi|_\mathfrak{a} - \{0\} \end{aligned}$$

Consider any subset $F \subset \Psi_0$. Define

$$\begin{aligned} \mathfrak{a}_F &= \{H \in \mathfrak{a} \ : \ \mu(H) = 0 \ \forall \mu \in F\} \\ \mathfrak{m}_F &= \{X \in \mathfrak{g} \ : \ [X, \mathfrak{a}_F] = 0\} \end{aligned}$$

The corresponding subgroups of G are given by

$$\begin{aligned} A_F &= \exp \mathfrak{a}_F \\ M_F &= \{g \in G \ : \ Ad(g)H = H \ \forall H \in \mathfrak{a}_F\} \end{aligned}$$

M_F is a real reductive group, and A_F is its split component. Further, define

$$\begin{aligned} \triangle_F^+ &= \{\alpha \in \triangle_0^+ \ : \ \alpha|_{\mathfrak{a}_F} \neq 0\} \\ \mathfrak{n}_F &= \sum_{\mu \in \triangle_F^+} \mathfrak{g}^\mu \\ N_F &= \text{connected subgroup of } G \text{ corresponding to } \mathfrak{n}_F \end{aligned}$$

$P_F = M_F N_F$ is a *standard parabolic subgroup* of G. (P_F, A_F) is a *standard parabolic pair* or *standard p-pair*.

Theorem 3.0.2 *The map $M_F \times N_F \to P_F$, $(m, n) \mapsto mn$, is a surjective diffeomorphism.*

From Section 1.3, we have $M_F = {}^0M_F A_F$.

Theorem 3.0.3 (Langlands decomposition) *The map* ${}^0M_F \times A_F \times N_F \to P_F$*,* $(m, a, n) \mapsto man$*, is a surjective diffeomorphism.*

Definition 3.0.4 *Let* P *be a subgroup of* G*. Then* P *is a parabolic subgroup if*
1. P *is its own normaliser in* G*.*
2. $\mathfrak{p}_{\mathbb{C}}$ *contains a Borel subalgebra of* $\mathfrak{g}_{\mathbb{C}}$*.*

Let N be the unipotent radical of P, $M = P \cap \theta P$, and A a split component of M. Then $M = {}^0M \cdot A$ and $P = MN$. (P, A) is a *parabolic pair* or *p-pair*. Each p-pair is conjugate in G to a unique standard p-pair.

Let P be a *minimal* parabolic subgroup. Then any p-pair (P, A) is conjugate to the standard p-pair $(P_\emptyset, A_\emptyset)$. And ${}^0M_\emptyset = M_\emptyset \cap K$, while ${}^0\mathfrak{m}_\emptyset = \mathfrak{m}_\emptyset \cap \mathfrak{k} = \mathfrak{m}_c$.

A parabolic subgroup P is called *maximal* if it is maximal among the proper parabolic subgroups. Then any p-pair (P, A) is conjugate to a standard p-pair (P_F, A_F), with $F = \Psi_0 - \{\mu\}$ for some $\mu \in \Psi_0$.

A standard p-pair (P_F, A_F) is *cuspidal* if ${}^0\mathfrak{m}_F$ has a Cartan subalgebra $\mathfrak{t}_F$ completely contained in $\mathfrak{k}$. Then $\mathfrak{h}_F = \mathfrak{a}_F + \mathfrak{t}_F$ is a *θ-stable* Cartan subalgebra of $\mathfrak{g}$. And every Cartan subalgebra of $\mathfrak{g}$ is conjugate to one such $\mathfrak{h}_F$ by an inner automorphism.

Finally, suppose that $F \subset F' \subset \Psi_0$. Then $A_F \supset A_{F'}$, $M_F \subset M_{F'}$, $N_F \supset N_{F'}$, and $P_F \subset P_{F'}$.

3.1 Bruhat decomposition

Let G be of inner type. Let $(P, A) = (P_\emptyset, A_\emptyset)$ be a minimal p-pair. Let $N_G(A) = \{g \in G \ : \ Ad(g)\mathfrak{a} = \mathfrak{a}\}$. Define $W(G, A) = N_G(A)/M = N_K(A)/{}^0M$ and view it as a group of linear automorphisms of $\mathfrak{a}$. If G is of inner type, then $W(G, A) = W(\mathfrak{g}, \mathfrak{a})$. Each $\tilde{s} \in W(G, A)$ has a representative $s \in K$.

Let $\tilde{s_*} \in W$ be the element that takes the positive roots to the negative roots.

Theorem 3.1.1 *G is the disjoint union of the sets PsP, $\tilde{s} \in W(G, A)$. Further, the term Ps_*P is an open submanifold of G and the other terms are submanifolds of lower dimension.* □

See [1] for details. Now let $\bar{N}$ denote the "opposite" nilpotent corresponding to the negative roots.

Corollary 3.1.2 *$\bar{N}P$ is an open dense submanifold of G.*

Proof. In the Bruhat decomposition, a double coset of $P = MN$ is

$$MNs\,P = Ns\,P = s(s^{-1}Ns)P \tag{1}$$

a translate of $(s^{-1}Ns)P$. For $s = s_*$, (1) is an open and dense submanifold of G, hence so is $(s^{-1}Ns)P = \bar{N}P$. □

Corollary 3.1.3 *Now let* (P_F, A_F) *be any p-pair. Then* $\bar{N}_F P_F$ *is an open dense submanifold of* G.

Proof. $\bar{N}_F P_F$ is open because $\mathfrak{g} = \bar{\mathfrak{n}}_F \oplus \mathfrak{p}_F$. Further,

$$P s_* P = s_* \bar{N}\, P \subset s_*(M_F \bar{N}_F) P_F = s_* \bar{N}_F P_F$$

□

4 $SL(n, \mathbb{R})$

Here $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the real $n \times n$ matrices with zero trace. $\mathfrak{k}$ consists of the skew-symmetric matrices, while $\mathfrak{p}$ consists of symmetric matrices. A choice of maximal abelian $\mathfrak{a}$ in $\mathfrak{p}$ is the set of diagonal matrices. Let $\mathfrak{m}$ be the centraliser of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{m} \cap \mathfrak{k} = 0$ and so $\mathfrak{h} = \mathfrak{a}$. Let e_i denote the member of $\mathfrak{a}^*$ which takes a diagonal matrix to its ith entry, and E_{ij} the matrix with all entries zero except for a 1 in the (i,j) position. Then

$$\triangle_0 = \{e_i - e_j : \ i \neq j\}, \text{ with } \mathfrak{g}^{e_i - e_j} = \mathbb{R} \cdot E_{ij}$$

Choose H_1 to be a diagonal matrix with entries $(a_1, \ldots, a_n)$ s.t. $a_1 > a_2 > \cdots > a_n$ and $a_1 + \cdots a_n = 0$. This gives the following choice of positive and simple roots:

$$\triangle_0^+ = \{e_i - e_j : \ i < j\}, \ \Psi_0 = \{\mu_i = e_i - e_{i+1} : \ i = 1, \ldots, n-1\}$$

Let $F \subset \Psi_0$ be obtained by deleting the simple roots $\mu_{n_1}, \ldots, \mu_{n_r}$, with $n_1 < \cdots < n_r$. Then $\mathfrak{a}_F$ consists of the diagonal matrices with entries $(a_1, \ldots, a_n)$, such that $a_1 = \cdots = a_{n_1}$, $a_{n_1+1} = \cdots = a_{n_2}$, $\ldots$, and $a_{n_r+1} = \cdots = a_n$. $\mathfrak{m}_F$ consists of block diagonal matrices, with block sizes $n_1, n_2 - n_1, \ldots, n - n_r$. $\mathfrak{n}_F$ consists of all the positive root spaces not already in $\mathfrak{m}_F$, i.e. the matrices with arbitrary entries on the upper right side of $\mathfrak{m}_F$, and 0's elsewhere. At the group level, we have the following picture:

$$P_F = \begin{pmatrix} \boxed{M_1} & & & * \\ & \boxed{M_2} & & \\ & & \ddots & \\ 0 & & & \boxed{M_k} \end{pmatrix}$$

where $k = n_r + 1$. M_F consists of the blocks M_i of size $n_i - n_{i-1}$. A_F consists of diagonal matrices with positive entries, whose intersection with each M_i is a scalar matrix. N_F has arbitrary values on the upper right of M_F, 1's on the diagonal, and 0's elsewhere.

${}^0M_F$ consists of blocks M_i each with determinant ± 1. This is a disconnected group, and its identity component consists of blocks with determinant 1. So ${}^0\mathfrak{m}_F$ consists of blocks with zero trace.

Consider the minimal p-pair $(P_\emptyset, A_\emptyset)$. $M_\emptyset$ is the set of diagonal matrices and $N_\emptyset$ is the set of upper triangular matrices with 1's on the diagonal. Further $A_\emptyset$ consists of the diagonal matrices with positive entries, while ${}^0M_\emptyset$ is the diagonal matrices with entries ± 1.

As for cuspidal parabolics, note that ${}^0\mathfrak{m}_F$ is a direct sum of $\mathfrak{sl}(m,\mathbb{R})$'s and that $\mathfrak{sl}(m,\mathbb{R})$ has a compact Cartan subalgebra[2] only if $m = 1, 2$. Hence cuspidal parabolics correspond to F's that have no consecutive members of Ψ_0, so that M_F has blocks of sizes 1 and 2 alone.

A Compact Cartan Subalgebras

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Every such $\mathfrak{h}$ has the same dimension and this common dimension is called the *rank* of $\mathfrak{g}$ (or G). Let K be the maximal compact subgroup of G corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{k}$ is a reductive Lie algebra and we can also define $\operatorname{rank}(K) = \operatorname{rank}(\mathfrak{k})$.

$\mathfrak{h}$ is a *compact* Cartan subalgebra if it is contained in $\mathfrak{k}$. Such an $\mathfrak{h}$ exists iff $\operatorname{rank}(G) = \operatorname{rank}(K)$. Now consider the case $G = SL(n,\mathbb{R})$, with $K = SO(n)$. Then $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(n,\mathbb{R})$. A Cartan subalgebra for $\mathfrak{g}$ is $\mathfrak{h} =$ diagonal matrices. A Cartan subalgebra for $\mathfrak{k}$ is $\mathfrak{t} =$ block diagonal matrices where each block is 2×2 and skew-symmetric. When n is odd, the last diagonal entry is 1. Now it is easily seen that $\operatorname{rank}(G) = n - 1$, while $\operatorname{rank}(K) = [n/2]$, and so $\operatorname{rank}(G) = \operatorname{rank}(K)$ iff $n = 1, 2$.

As we shall see later, $\operatorname{rank}(G) = \operatorname{rank}(K)$ is a case of great importance. This is exactly the condition for G to have a discrete series.

References

[1] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1978.

[2] A. W. Knapp. *Representation Theory of Semisimple Groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1986.

[3] N. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1988.

[2] See Appendix