Representations of Real Reductive Groups - V

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1 Introduction

In this lecture we work out some of the representation theory of $SL(2,\mathbb{C})$. The algebraic approach is more complicated here than for $SL(2,\mathbb{R})$. This is partly due to the doubling of dimension, and otherwise to the fact that the maximal compact subgroup $K = SU(2)$ is not abelian. Somewhat paradoxically, the eventual classification of irreducible unitary representations turns out to be simpler.

Let us recall how we tackled $SL(2,\mathbb{R})$. First, we used the algebraic approach to classify the irreducible $(\mathfrak{g}, K)$-modules. This involved writing down what the root spaces of $\mathfrak{g}_{\mathbb{C}}$ would do to the K-types. To find the unitary subset, we wrote down the conditions for the elements of $\mathfrak{g}$ to act as skew-Hermitian operators. Finally, we used the analytic method of induction to construct the principal series, and to see that all the irreducible representations are subquotients of these.

The same approach could be used for $SL(2,\mathbb{C})$. This is done explicitly in Taylor[2]. However, the algebraic approach involves a fair amount of manipulation and takes more space than is appropriate for this lecture, so we avoid it. Further, since the analytic approach got rather short shrift in the $SL(2,\mathbb{R})$ lecture, this is a good opportunity to remedy matters.

The Iwasawa decomposition of G is given by

$$\begin{aligned} K &= SU(2) \\ A &= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \\ N &= \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},\ z \in \mathbb{C} \end{aligned}$$

2 Induction

This starts off as a repeat of Appendix B of Lecture II, with some minor differences. The new material starts with the discussion of the "noncompact picture".

Let B be the *Borel subgroup* of upper-triangular matrices. Corresponding to the Iwasawa decomposition $G = KAN$, there is the *Levy-Langlands decomposition* $B = {}^oMAN$, where ${}^oM$ = diagonal subgroup of $K \cong S^1$. Let χ be a quasi-character of B. Then $\chi = (\xi, \eta)$, where ξ is a character of ${}^oM$, and $\eta : A \to \mathbb{C}^\times$ is a quasicharacter of A. N acts trivially. Let $n \in \mathbb{Z}$ represent ξ, with

$$\xi \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{in\theta}$$

And the quasi-character η corresponds to a $\lambda \in \mathbb{C}$:

$$\eta \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = e^{\lambda t}$$

Let $X = G/B$, with measure μ obtained from the Haar measure on G. This measure is not G-invariant, but we have (as for all Lie groups):

$$d\mu_x/d\mu(g) = \rho_x(g) \in L^\infty_{loc}(X)$$

where $\mu_x(\mathcal{B}) = \mu(x^{-1} \cdot \mathcal{B})$. For $SL(2, \mathbb{C})$ we have the formula [3, p 82-83]

$$\rho_x(g)^{1/2} = \delta(a(x^{-1}g))^{-1}$$

with $\delta \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = e^{2t}$, and $a()$ the projection onto A.

Now we can describe $\pi_\chi = \pi_{n,\lambda}$, the representation of G induced from χ. The representation space is

$$H_\chi = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(gb) = \chi(b^{-1}) f(g) \ \forall g \in G, b \in B \\ \int_X ||f([g])||^2 d\mu([g]) < \infty \end{array} \right\}$$

with inner product $\langle f_1, f_2 \rangle = \int_X \langle f_1([g]), f_2([g]) \rangle d\mu([g])$. The G action is

$$(x \cdot f)(g) = \rho_x(g)^{1/2} f(x^{-1}g)$$

The $\rho_x^{1/2}$ factor keeps the representation unitary when χ is unitary.

2.1 Compact picture

Note that $X \cong K/{}^oM$. We can now represent H as functions on K with an invariance property wrt ${}^oM$. Then, just as for $SL(2, \mathbb{R})$,

$$H_\chi \cong H^n = \left\{ f \in L^2(K) |\ f(km) = \xi(m)^{-1} f(k), k \in K,\ m \in {}^oM \right\}$$

The identification of H_χ with H^n is given by restriction to K. Calculating as for $SL(2, \mathbb{R})$, (see App B, Lec II), we get the action

$$(x \cdot f)(u) = e^{-(\lambda+2)t(x^{-1}u)} f(x^{-1}[u])$$

It was the compact picture that we had used in Lecture II to connect the analytic constructions with the algebraic classification, and thus to show the reducibility of the unitary principal series. In this lecture, we seek to show irreducibility directly, and for this it is more convenient to use a different description, which we now give.

2.2 Noncompact picture

Instead of restricting the induced picture to K, we now restrict it to $\bar{N} = N^t$. The motivation is that $\bar{N}B$ is an open dense submanifold of G — it consists of the matrices whose upper left entry is non-zero. So we can restrict first to $\bar{N}B$ and then to $\bar{N}$ without loss of information.

The representation space gets identified with $L^2(\bar{N})$, since $\bar{N} \cap K = \bar{N} \cap B = I$. The identification is given in one direction by restriction to $\bar{N}$. In the other direction $F \in L^2(\bar{N})$ corresponds to $f \in H_\chi$ defined by

$$f(\bar{n}b) = \chi(b)^{-1} F(\bar{n})$$

on $\bar{N}B$. To get the G action on $F \in L^2(\bar{N})$, we first let G act on f, then restrict the result to $\bar{N}B$. Let us write $[\bar{n}b] = \bar{n}$. This gives

$$(x \cdot F)(\bar{n}) = (x \cdot f)(\bar{n}) = \rho_x(\bar{n})^{1/2} f(x^{-1}\bar{n}) = \rho_x(\bar{n})^{1/2} \chi(b(x^{-1}\bar{n}))^{-1} F([x^{-1}\bar{n}])$$

This can be made very explicit. For an element of $\bar{N}B$, the decomposition[1] is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

Let us identify $\bar{N}$ with $\mathbb{C}$ in the obvious manner. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} d - bz & -b \\ az - c & a \end{pmatrix}$$

$$\implies [x^{-1}\bar{n}] = \frac{az - c}{d - bz}, \text{ where } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \bar{n} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

We also calculate the ${}^oMAN$ decomposition

$$\begin{aligned} b(x^{-1}\bar{n}) &= \begin{pmatrix} d - bz & -b \\ 0 & (d - bz)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{d-bz}{|d-bz|} & 0 \\ 0 & (\frac{d-bz}{|d-bz|})^{-1} \end{pmatrix} \begin{pmatrix} |d - bz| & 0 \\ 0 & |d - bz|^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This gives

$$\chi(b(x^{-1}\bar{n}))^{-1} = \left(\frac{d - bz}{|d - bz|}\right)^{-n} |d - bz|^{-\lambda}, \ \rho_x(\bar{n})^{1/2} = |d - bz|^{-2}$$

The G action on $L^2(\bar{N}) = L^2(\mathbb{C})$ can now be written out:

$$\pi_{n,\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(z) = |d - bz|^{n-2-\lambda} (d - bz)^{-n} F\left(\frac{az - c}{d - bz}\right)$$

[1]Note the unfortunate use of b for both a matrix and a matrix entry.

Theorem 2.3 *The representations $\pi_{n,i\nu}$, $\nu \in \mathbb{R}$, are all irreducible and unitary.*

Proof. $\nu \in \mathbb{R}$ implies that χ is a unitary representation of B, and so $\pi_{n,i\nu}$ is a unitary representation of $SL(2,\mathbb{C})$. Now suppose there is a subrepresentation. Then projection onto the invariant subspace is a bounded operator which commutes with all the $\pi_{n,\lambda}(x)$. We will show that any operator R which commutes with all $\pi_{n,\lambda}(b)$, $b \in B^t$, is a scalar, so that $\pi_{n,\lambda}$ is in fact irreducible on B^t. Let us use the noncompact picture.

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \Longrightarrow (b_1 \cdot F)(z) = F(z+u) \\ b_2 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Longrightarrow (b_2 \cdot F)(z) = |a|^{2+\lambda-n} a^n F(a^2 z) \end{aligned}$$

Consider the Fourier transform on $L^2(\mathbb{C})$:

$$\hat{F}(w) = \frac{1}{2\pi} \int F(z) e^{-i\mathrm{Re}(\bar{z}w)}\, dx dy$$

$$(b_1 \cdot F)\hat{}(w) = e^{i\mathrm{Re}(\bar{u}w)} \hat{F}(w) \tag{1}$$

$$(b_2 \cdot F)\hat{}(w) = |a|^{2+\lambda-n} a^n \hat{F}(\bar{a}^{-2} w) \tag{2}$$

Define the operator $\hat{R}$ by $\hat{R}\hat{F} = (RF)\hat{}$. Now formula (1) implies $\hat{R}$ is multiplication by a bounded (Borel) function r; and then formula (2) shows that $r(w) = r(\bar{a}^{-2}w)$ for almost all w. Hence r is constant almost everywhere, and R is a scalar. □

The representations $\pi_{n,i\nu}$, $\nu \in \mathbb{R}$, are called the *unitary principal series* of $SL(2,\mathbb{C})$. The full *principal series* consists of all the $\pi_{n,\lambda}$.

3 Complementary Series

Recall that for $SL(2,\mathbb{R})$ the complementary series representations were isomorphic[2] to the principal series representations $\pi_{0,\lambda}$, $\lambda \in (0,1)$, but with a different norm. Similarly, we can obtain a complementary series for $SL(2,\mathbb{C})$. Consider the noncompact picture for $\pi_{0,\lambda}$, $\lambda \in (0,2)$, and use the new inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(z)\overline{g(\zeta)}}{|z-\zeta|^{2-\lambda}}\, dz\, d\zeta$$

The resulting representations are irreducible and unitary. We leave alone for now the question as to how one comes up with this particular series.

[2] At the $(\mathfrak{g}, K)$ level.

4 Last Words

Theorem 4.1 *The unitary principal series and the complementary series yield all the non-trivial irreducible unitary representations of $SL(2, \mathbb{C})$.* □

One proof of this follows the same lines as the classification we carried out for $SL(2, \mathbb{R})$. We can first classify the irreducible $(\mathfrak{g}, K)$-modules and then find out which ones are unitary, and finally match that list to the members of the unitary principal and complementary series. A reference for this is Taylor[2]. Alternatively, we wait till we have developed further theory, such as the definition of characters and the Langlands classification, and then this special case drops out easily. This is the approach taken in Knapp[1].

The analytic techniques used in this lecture will be at the heart of the developments in the next two lectures. The upper triangular matrices are replaced by the more general parabolic subgroups. Recall the fact that $\bar{N}B$ was dense in G. This is true in general and allows the use of the noncompact picture to investigate irreducibility of the principal series. In common with $SL(2, \mathbb{C})$, all complex semisimple groups have irreducible unitary principal series. While for a general real reductive group we will see that a generic unitary principal series representation is irreducible.

References

[1] A. W. Knapp. *Representation Theory of Semisimple Groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1986.

[2] M. E. Taylor. *Noncommutative Harmonic Analysis*, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 1986.

[3] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, volume 16 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1989.