Representations of Real Reductive Groups - IV

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1 Introduction

In this lecture we investigate the centre of the enveloping Lie algebra and prove the Harish-Chandra isomorphism theorem.

2 Reductive Lie Algebras

A Lie algebra $\mathfrak{g}$ is *reductive* if $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. Now $\mathfrak{g}_1$ has a Cartan subalgebra $\mathfrak{h}_1$ and a root decomposition

$$\mathfrak{g}_1 = \mathfrak{h}_1 + \sum_{\alpha \in \triangle(\mathfrak{g}_1, \mathfrak{h}_1)} \mathfrak{g}_\alpha$$

The decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1$ implies that $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{g}$. The resulting root space decomposition of $\mathfrak{g}$ is simply

$$\mathfrak{g} = \mathfrak{h} + \sum_{\tilde{\alpha} \in \triangle(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{\tilde{\alpha}}$$

with $\triangle = \triangle(\mathfrak{g}, \mathfrak{h}) \cong \triangle(\mathfrak{g}_1, \mathfrak{h}_1)$. The identification of the two root systems is as follows: extend $\alpha \in \triangle(\mathfrak{g}_1, \mathfrak{h}_1) \subset \mathfrak{h}_1^*$ to $\tilde{\alpha} \in \mathfrak{h}^*$ by requiring $\tilde{\alpha}(\mathfrak{z}(\mathfrak{g})) = 0$. Then $\mathfrak{g}_\alpha = \mathfrak{g}_{\tilde{\alpha}}$. The Weyl group $W(\triangle)$ acts trivially on $\mathfrak{z}(\mathfrak{g})$.

Consequently, arguments about reductive Lie algebras generally reduce to arguments about semisimple Lie algebras. Instead of stating the appropriate reduction in each situation that comes up, we will assume in the rest of this lecture that $\mathfrak{g}$ is semisimple, and note that all the theorems obtained are also true for the reductive case.

3 Chevalley Reduction Theorem

For a finite dimensional vector space V, $\mathcal{P}(V) = \mathcal{S}(V^*)$ is the algebra of *polynomial functions* on V.

Let $G = Int(\mathfrak{g})$. It acts on $\mathcal{P}(\mathfrak{g})$ by $(\sigma f)(X) = f(\sigma^{-1} X)$. Let $\mathcal{P}(\mathfrak{g})^G$ denote the elements of $\mathcal{P}(\mathfrak{g})$ which are fixed by G.

Now consider $\mathcal{P}(\mathfrak{h})$. W acts on $\mathfrak{h}^*$ and hence on $\mathcal{P}(\mathfrak{h})$. Let $\mathcal{P}(\mathfrak{h})^W$ be the polynomial functions fixed by W.

Any polynomial function on $\mathfrak{g}$, when restricted to $\mathfrak{h}$, is a polynomial function on $\mathfrak{h}$. So we have the restriction map $\theta : \mathcal{P}(\mathfrak{g}) \to \mathcal{P}(\mathfrak{h})$.

Theorem 3.1 (Chevalley) *θ is an algebra isomorphism of $\mathcal{P}(\mathfrak{g})^G$ onto $\mathcal{P}(\mathfrak{h})^W$.*

Proof. Suppose $f \in \mathcal{P}(\mathfrak{g})^G$. Then it is fixed by the inner automorphism $\tau_\alpha = \exp ad(x_\alpha) \exp ad(-y_\alpha) \exp ad(x_\alpha)$, and so $\theta(f) = f|_{\mathfrak{h}}$ is fixed by $\tau_\alpha|_{\mathfrak{h}} = \sigma_\alpha$. So $\theta(f)$ is fixed by W.

We skip the proof of surjectivity. It consists of explicit constructions of elements of $\mathcal{P}(\mathfrak{g})^G$ which map onto elements $sym(\lambda^k)$, $\lambda \in \Lambda^+$, and these span $\mathcal{P}(\mathfrak{h})^W$. [1]

Now suppose $f \in \mathcal{P}(\mathfrak{g})^G$ such that $f|_{\mathfrak{h}} = 0$. Then $f = 0$ on all the images of $\mathfrak{h}$ under G, i.e. on all the Cartan subalgebras. Therefore $f = 0$ on the regular semisimple elements of $\mathfrak{g}$ (as each of them lies in a Cartan subalgebra). But these are dense in $\mathfrak{g}$,[2] and hence $f = 0$ on $\mathfrak{g}$. $\square$

Recall that $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$, where $G = Int(\mathfrak{g})$. Also, PBW implies $\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$ (as linear spaces). This is actually an isomorphism of G-modules, and so $\mathcal{U}(\mathfrak{g})^G \cong \mathcal{S}(\mathfrak{g})^G$. The Killing form identifies $\mathfrak{g}$ and $\mathfrak{h}$ with $\mathfrak{g}^*$ and $\mathfrak{h}^*$, hence $\mathcal{S}(\mathfrak{g})^G$ with $\mathcal{P}(\mathfrak{g})^G$, and $\mathcal{U}(\mathfrak{h})^W \cong \mathcal{S}(\mathfrak{h})^W$ with $\mathcal{P}(\mathfrak{h})^W$. So the reduction theorem leads to the following sequence: $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G \longrightarrow \mathcal{S}(\mathfrak{g})^G \longrightarrow \mathcal{P}(\mathfrak{g})^G \xrightarrow{\theta} \mathcal{P}(\mathfrak{h})^W$, where all the maps are linear isomorphisms, and the last two are also algebra isomorphisms.

Let us look at the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. As usual, consider the standard basis H, X, Y, as defined in Lecture II. Take the Cartan subalgebra to be $\mathfrak{h} = \mathbb{C} \cdot H$. Now we can use the reduction theorem to describe $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$.

Corollary 3.2 *$\mathcal{Z}(\mathfrak{sl}(2, \mathbb{C})) = \mathbb{C}[\Omega]$, where $\Omega = 1 + H^2 + 2(XY + YX)$.*

Proof. . Ω corresponds to the polynomial ω on $\mathfrak{g}$ given by the same formula. And then $\theta(\omega) = 1 + H^2$. The Weyl group W here consists of two elements and the non-trivial one maps $H \mapsto -H$. Therefore $\mathcal{P}(\mathfrak{h})^W$ is generated by H^2 — or $1 + H^2$. Since the last two maps in the sequence were algebra isomorphisms, this means $\mathcal{S}(\mathfrak{g})^G$ is generated by ω. Now consider $Z \in \mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$. This is the image of some $p(\omega) \in \mathbb{C}[\omega] = \mathcal{S}(\mathfrak{g})^G$. Let $p(\omega) = a_o\omega^k + a_1\omega^{k-1} + \cdots a_k$. Then $Z = a_o\Omega^k + Z_1$ where Z_1 is of lower degree than Z. Repeat the process with Z_1. By induction we see that Z is a polynomial in Ω. $\square$

4 Harish-Chandra Isomorphism

Here we describe a different characterisation of the centre $\mathcal{Z}$. In a sense that the discussion below will make clearer, this is a perturbation of the description

[1] For the specifics see [2].

[2] See Appendix.

given by the Chevalley reduction theorem. The defect we seek to correct is that use of that theorem involves the identification $\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$, which is not an algebra homomorphism.

Consider the projection from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{h})$ given by the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^{-}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n})$ and let p be its restriction to $\mathcal{Z}$. Then p can be shown to be an algebra homomorphism. Further, the image of p lies in $\mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})n$.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and choose the positive root to be given by $\alpha(H) = 2$. Look at Ω again. We can express it also as $\Omega = (H+1)^2 + 4YX = (H-1)^2 + 4XY$. Then $p(\Omega) = (H+1)^2$. Now Ω is an invariant of the algebra[3], but this image $p(\Omega)$ is not. Of course, it must depend on the choice of $\mathfrak{h}$, but we see it also depends on the choice of the positive root: the other choice would give the image $(H-1)^2$. So we are led to a correcting factor. Define $u(H) = H - \rho(H) \cdot 1$, where ρ is half the sum of the positive roots, and extend to a homomorphism $u : \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$. Let $\gamma = u \circ p$. Then $\gamma(\Omega) = H^2$ and this is independent of the choice of positive root. Another way of saying this is that it is invariant under the Weyl group.

Let's return to the general set-up. Now $u : \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$ is defined by extending from the expression $u(H) = H - \rho(H) \cdot 1$, $\forall H \in \mathfrak{h}$. Let $\gamma = u \circ p$.

Theorem 4.1 (Harish-Chandra) *γ is injective on $\mathcal{Z}$ and in fact is an algebra isomorphism of $\mathcal{Z}$ onto $\mathcal{U}(\mathfrak{h})^W$.*

Proof. We write only a general description of the proof. For precise details, one may look at either of the references below. First, let's see why γ maps $\mathcal{Z}$ into $\mathcal{U}(\mathfrak{h})^W$. It is enough to show that the image is invariant under the simple reflections σ_α, $\alpha \in \Psi$. Acting on the image by σ_α has the same result as starting with the base $\sigma_\alpha\Psi$ instead of Ψ. The resulting change in p is now exactly corrected by the change in ρ.

The identifications described in the previous section lead to the following diagram. If it commutes, we are done.

$$\begin{tikzcd}
\mathcal{S}(\mathfrak{g})^G \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{Z} \arrow[r, "\gamma"] & \mathcal{U}(\mathfrak{h})^W \arrow[d, leftrightarrow] \\
\mathcal{P}(\mathfrak{g})^G \arrow[rr, "\theta"] & & \mathcal{P}(\mathfrak{h})^W
\end{tikzcd}$$

Consider the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let us start with $\Omega \in \mathcal{Z}$, which maps to $H^2 \in \mathcal{U}(\mathfrak{h})^W$, and then to $H^{*2} \in \mathcal{P}(\mathfrak{h})^W$. In the other direction, Ω corresponds to $H^2 + 1 + 2XY + 2YX \in \mathcal{S}(\mathfrak{g})^G$ and hence to $H^{*2} + 1 + 2X^*Y^* + 2Y^*X^* \in \mathcal{P}(\mathfrak{g})^G$. Under θ that maps to $H^{*2} + 1 \in \mathcal{P}(\mathfrak{h})^W$.

So the diagram is not commutative. On the other hand, we see that the failure is measured by a term of lower degree. This holds in general, by an

[3] It equals 1 + a multiple of the Casimir element.

argument analogous to the one used in Corollary 3.2. Therefore we can push the argument for surjectivity through by induction on degree. □

5 Infinitesimal Characters

In Lecture I we saw that $\mathcal{Z}$ acts by scalars on any irreducible $(\mathfrak{g}, K)$-module V. That is, $\exists$ homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$ s.t. $Z \cdot v = \chi(Z)v\ \forall v \in V,\ Z \in \mathcal{Z}$. Now we seek to characterise these *infinitesimal characters* χ via the Harish-Chandra isomorphism.

Let $\lambda \in \mathfrak{h}^*$. It extends to a homomorphism from $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. Define $\chi_\lambda(Z) = \lambda(\gamma(Z))$. This is a homomorphism from $\mathcal{Z}$ into $\mathbb{C}$. Let $w \in W$. Then $w\gamma = \gamma \ \Rightarrow\ \chi_{w\lambda} = \chi_\lambda$.

Theorem 5.1 *Let $\lambda, \mu \in \mathfrak{h}^*$ and $\chi : \mathcal{Z} \to \mathbb{C}$ be a homomorphism. Then:*
(1) $\chi_\lambda = \chi_\mu$ iff λ and μ are in the same W-orbit.
(2) $\exists\, \lambda \in \mathfrak{h}^*$ s.t. $\chi = \chi_\lambda$.

Proof. (1) Suppose λ and μ are not in the same W-orbit. Then we can find a polynomial P on $\mathfrak{h}^*$ which is non-zero on λ, and zero on all the other W-conjugates of λ as well as on all the W-conjugates of μ. This P can be identified with an element of $\mathcal{U}(\mathfrak{h}) \cong \mathcal{S}(\mathfrak{h})$, while elements of $\mathfrak{h}^*$ extend to homomorphisms from $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. So we can define $\lambda(P)$ and it turns out to equal $P(\lambda)$. Further, the two W-actions are related by $(w\lambda)(P) = \lambda(w^{-1}P)$. The relations between P, λ, and μ, can now be restated: λ is non-zero on P and zero on all its other W-conjugates, while μ is zero on all the W-conjugates of P. Define $Q = \sum_{w\in W} wP$. This is an element of $\mathcal{U}(\mathfrak{h})^W$ on which λ is non-zero but μ is zero. Let Z be the element of $\mathcal{Z}$ corresponding to Q. Then $\chi_\lambda(z) \neq 0 = \chi_\mu(z)$.

(2) Consider $\chi : \mathcal{U}(\mathfrak{h})^W \cong \mathcal{Z} \to \mathbb{C}$. $\mathcal{U}(\mathfrak{h})$ is integral over $\mathcal{U}(\mathfrak{h})^W$, so every complex homomorphism of $\mathcal{U}(\mathfrak{h})^W$ extends to $\mathcal{U}(\mathfrak{h})$. Let χ' be the extension of χ. Let $\lambda = \chi'|_{\mathfrak{h}}$. Then $\chi = \chi_\lambda$. □

Example Let $V(\lambda)$ be the irreducible cyclic $\mathfrak{g}$-module with highest weight λ. Let $v \in V_\lambda$ be a highest weight vector. Let $Z \in \mathcal{Z}$. Then $Z = p(Z) + X$, where $X \in \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Note also that $\lambda \circ p = (\lambda + \rho) \circ \gamma$. This yields

$$Z \cdot v = (p(Z) + X) \cdot v = p(Z) \cdot v = \lambda(p(Z))v = (\lambda + \rho)(\gamma(Z))v$$

So that the infinitesimal character of this representation is $\chi_{\lambda+\rho}$.[4]

[4] For this reason, some authors prefer to use χ_λ for the infinitesimal character corresponding to $\lambda + \rho$.

A Regular Elements

Consider the characteristic polynomial of $ad(X)$, $X \in \mathfrak{g}$. We write

$$\det(tI - ad(X)) = \sum_{j=0}^{n} t^j D_j(X)$$

where D_j is a polynomial on $\mathfrak{g}$ and $n = \dim \mathfrak{g}$. Let r be the smallest index s.t. D_r is not identically zero. We say X is *regular* if $D_r(X) \neq 0$. Let $\mathcal{R}$ denote the set of regular elements. $\mathcal{R}$ is Zariski-open and hence dense.

Theorem A.1 *$X \in \mathcal{R} \Rightarrow ad(X)$ is semisimple. Further, the centraliser in $\mathfrak{g}$ of $X \in \mathcal{R}$ is a Cartan subalgebra.*

References

[1] Harish-Chandra. On some applications of the universal enveloping algebra of a semisimple Lie algebra. *Trans. Amer. Math. Soc.*, 70:28–96, 1951.

[2] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1972.

[3] N. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, 1988.