Representations of Real Reductive Groups - III

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1 Introduction

In this lecture we collect some common knowledge about the universal enveloping algebra and semisimple Lie algebras. The aim is to collect just enough so that in the next lecture we can prove the Harish-Chandra isomorphism theorem for the centre of the universal enveloping algebra.

As usual, $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$.

2 $\mathcal{U}(\mathfrak{g})$

Think of $\mathfrak{g}$ as a vector space and let $\mathcal{T} = \mathcal{T}(\mathfrak{g})$ be its tensor algebra. $\mathcal{I}$ is the two sided ideal generated by $\{XY - YX - [X,Y] : X, Y \in \mathfrak{g}\}$. Then the *universal enveloping algebra* of $\mathfrak{g}$ is defined to be $\mathcal{U}(\mathfrak{g}) = \mathcal{T}/\mathcal{I}$. The sequence $\mathfrak{g} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{I}$ gives an embedding $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$. $\mathcal{U}(\mathfrak{g})$ is an associative algebra with 1, and has the following universal property: Let $\mathcal{A}$ be any associative algebra with 1. Give it a Lie algebra structure by setting $[a,b] = ab - ba$. Then any Lie algebra homomorphism $\mathfrak{g} \to \mathcal{A}$ extends to an algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{A}$.

Theorem 2.1 (Poincaré-Birkhoff-Witt) *Suppose $X_1, \ldots X_m$ is a basis of $\mathfrak{g}$. Then the monomials $X_1^{n_1} \cdots X_m^{n_m}$ form a basis of $\mathcal{U}(\mathfrak{g})$.* □

Let $\mathcal{S}(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. Consider the map $sym : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ defined by

$$sym(X_1 \cdots X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma 1} \cdots X_{\sigma k}$$

Then P-B-W can be interpreted in the following language.

Theorem 2.2 *$sym : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is a linear isomorphism.* □

Although sym is not an algebra isomorphism in general, it does satisfy the following relation. Let $P, Q \in \mathcal{S}(\mathfrak{g})$ be monomials of degree p, q. Then $sym(PQ) - sym(P)sym(Q)$ has degree $< p + q$. So sym preserves the term of highest degree.

3 Semisimple Lie Algebras

Our aim in this section is to paint an overall picture of the situation, while bringing out the details needed for the next lecture. So we make no attempt at arranging our statements in an order corresponding to any possible sequence of deductive proofs.

A Lie algebra is *simple* if it has no proper non-zero ideal. It is *semisimple* if it has no non-zero solvable ideal. The Killing form on a Lie algebra $\mathfrak{g}$ is defined by $(X, Y) = tr(ad(X)ad(Y))$.

Theorem 3.1 *A Lie algebra $\mathfrak{g}$ is semisimple iff one of the following holds:*
1. The Killing form is non-degenerate.
2. It is a Lie algebra direct sum of simple ideals. □

From here on $\mathfrak{g}$ is assumed to be semisimple. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is maximal among the (abelian) subalgebras consisting of semisimple elements. For a given $\mathfrak{h}$, there is a decomposition of $\mathfrak{g}$ into common eigenspaces of the elements of $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \triangle} \mathfrak{g}_\alpha$$

where $\triangle = \triangle(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$, $X \in \mathfrak{g}_\alpha$ if $ad(H)X = \alpha(H)X \ \forall H \in \mathfrak{h}$, and $\alpha \in \triangle$ iff $\mathfrak{g}_\alpha \neq 0$. If $\alpha \in \triangle$, then it is a *root* and $\mathfrak{g}_\alpha$ is a *root space*.

The root system $\triangle$ has subsets called *bases* distinguished by the following property: Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ be a base. Then any $\alpha \in \triangle$ satisfies $\alpha = \sum_{i=1}^n n_i\alpha_i$, with all n_i's non-negative or all n_i's non-positive. Thus, for a fixed base, we have a natural notion of *positive* and *negative* roots, denoted by $\triangle^\pm$. Further, $\alpha \in \triangle$ iff $-\alpha \in \triangle$, so that $\triangle^- = -\triangle^+$. The members of a base are called *simple* roots.

We put a partial order on $\mathfrak{h}^*$ by saying $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots.

The simplest of the simple Lie algebras is $\mathfrak{sl}(2, \mathbb{C})$. Recall from Lecture II the elements H, X, Y satisfying $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. Then $\mathfrak{h} = \mathbb{C} \cdot H$ is a Cartan subalgebra, and there are two roots $\pm\alpha$, with $\alpha(H) = 2$.

An important aspect of the structure of any semisimple $\mathfrak{g}$ is that it is a sum of copies of $\mathfrak{sl}(2, \mathbb{C})$. Let $\alpha \in \triangle$. Then $\exists\ H_\alpha \in \mathfrak{h}$, $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, s.t. $\text{span}\{H_\alpha, X_\alpha, Y_\alpha\} \cong \mathfrak{sl}(2, \mathbb{C})$ under the map $H_\alpha \mapsto H$, $X_\alpha \mapsto X$, $Y_\alpha \mapsto Y$. For each choice of Ψ this leads to a *Chevalley basis* for $\mathfrak{g}$: $\{X_\alpha, Y_\alpha : \alpha \in \triangle^+\} \cup \{H_{\alpha_i} : \alpha_i \in \Psi\}$.

The Killing form of $\mathfrak{g}$ on restriction to $\mathfrak{h}$ remains non-degenerate. So we have an identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ and an inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$. The identification matches $\alpha \in \triangle$ with a multiple of H_α.

Let P_α be the hyperplane in $\mathfrak{h}^*$ orthogonal to $\alpha \in \triangle$. Let σ_α be the reflection across this hyperplane. The group $W = W(\triangle)$ generated by these reflections is called the *Weyl group*. It preserves $\triangle$ (and hence is finite) and acts simply transitively on the set of bases. Further, W is in fact generated by $\{\sigma_\alpha | \alpha \in \Psi\}$.

Now consider a $\mathfrak{g}$ module V on which $\mathfrak{h}$ acts semisimply. Then there is a decomposition into common eigenspaces of $\mathfrak{h}$:

$$V = \sum_{\lambda \in \Lambda(V)} V_\lambda$$

where $\Lambda(V) \subset \mathfrak{h}^*$, $V_\lambda = \{v \in V | Hv = \lambda(H)v \; \forall H \in \mathfrak{h}\}$, and $\lambda \in \Lambda(V)$ iff $V_\lambda \neq 0$. The elements of $\Lambda(V)$ are called the *weights of* V. Note that $\mathfrak{g}_\alpha$ maps V_λ into $V_{\lambda+\alpha}$.

A special case of this is the *cyclic module of highest weight* λ. Such a module V has a vector $v \in V_\lambda$ s.t. $\mathfrak{g}_\alpha \cdot v = 0 \; \forall \alpha \in \Psi$, and $V = \mathcal{U}(\mathfrak{g}) \cdot v$. Here v is a *highest weight vector* and λ is a *highest weight*. For each $\lambda \in \mathfrak{h}^*$, there is a unique irreducible cyclic module of highest weight λ.

Theorem 3.2 *Let $V = V(\lambda)$ be the irreducible cyclic module of highest weight λ. Then $\lambda \neq \mu \in \Lambda(V)$ implies $\lambda > \mu$.* □

An element λ of $\mathfrak{h}^*$ is an *integral weight* if $\langle \lambda, \alpha \rangle = 2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z} \; \forall \alpha \in \triangle$. The set of integral weights is denoted Λ, forms a subgroup of $\mathfrak{h}^*$, and contains $\triangle$. The *root lattice* Λ_r is the subgroup of Λ generated by $\triangle$.

An integral weight λ is *dominant* if $\langle \lambda, \alpha \rangle \geq 0 \; \forall \alpha \in \Psi$. Λ^+ is the set of dominant integral weights. The W-orbit of any $\lambda \in \Lambda$ intersects Λ^+ exactly once. If $\lambda \in \Lambda^+$ then there are only finitely many $\mu \in \Lambda^+$ s.t. $\mu < \lambda$.

Theorem 3.3 *Let $V = V(\lambda)$ be the irreducible cyclic module of highest weight λ. It is finite-dimensional iff λ is a dominant integral weight. Then $\Lambda(V)$ is permuted by W and* $\dim(V_\mu) = \dim(V_{\sigma\mu}) \; \forall \sigma \in W$. □

4 Automorphisms of $\mathfrak{g}$

The group of *inner automorphisms*, $Int(\mathfrak{g})$, is generated by elements of the form $\exp ad(X)$, where $ad(X)$ is nilpotent. It is a normal subgroup of $Aut(\mathfrak{g})$, the full group of automorphisms of $\mathfrak{g}$.

Now consider the group $\Gamma(\mathfrak{g})$ of automorphisms of $\triangle$ which preserve the base Ψ. If $\mathfrak{g}$ is simple this group is non-trivial only for the cases $A_n (n \geq 2)$, $D_n (n \geq 4)$ and E_6. $\Gamma(\mathfrak{g})$ can be identified with a subgroup of $Aut(\mathfrak{g})$. Then we have a semi-direct product: $Aut(\mathfrak{g}) = Int(\mathfrak{g}) \cdot \Gamma(\mathfrak{g})$.

Theorem 4.1 *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Cartan subalgebras of $\mathfrak{g}$. Then $\exists \; \tau \in Int(\mathfrak{g})$ s.t. $\tau \mathfrak{h}_1 = \mathfrak{h}_2$.* □

5 $\mathcal{Z}(\mathfrak{g})$

Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ be the centre of $\mathcal{U}(\mathfrak{g})$. Let $G = Int(\mathfrak{g})$. The action of G on $\mathcal{U}(\mathfrak{g})$ restricts to $\mathcal{Z}$.

Theorem 5.1 $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$.

Proof. Consider a generator of G, $\exp ad(X)$, with $ad(X)$ nilpotent, and $Z \in \mathcal{Z}$. Then $ad(X)Z = [X, Z] = 0$ implies $\exp ad(X)(Z) = Z$. Since the generators of G fix Z, all of G does.

Now suppose G fixes $X \in \mathcal{U}(\mathfrak{g})$. Note that $\mathfrak{g}$ is generated by its root spaces, so it suffices to show that $[X, X_\alpha] = 0$ for each $X_\alpha \in \mathfrak{g}_\alpha$. Let $n = ad(X_\alpha)$ and t the largest integer s.t. $n^t \neq 0$. Choose distinct members $a_1, \ldots, a_{t+1}$ of $\mathbb{C}$. Then $\exp a_i n = 1 + a_i n + \cdots + (a_i/t!)n^t$ fixes X. Now

$$\begin{vmatrix} 1 & a_1 & a_1^2/2! & \cdots & a_1^t/t! \\ \vdots & & & & \vdots \\ 1 & a_{t+1} & a_{t+1}^2/2! & \cdots & a_{t+1}^t/t! \end{vmatrix} = (2!3!\ldots t!)^{-1} \prod_{i>j} (a_i - a_j) \neq 0$$

Therefore, the vectors $\exp a_i n$ have the same span as the vectors $1, \ldots, n^t$. In particular, $\exists$ scalars $b_1, \ldots, b_{t+1}$ s.t. $n = \sum_{i=1}^{t+1} b_i (\exp a_i n)$. And then

$$ad(X_\alpha)X = \sum_{i=1}^{t+1} b_i (\exp a_i n) X = (\sum_{i=1}^{t+1} b_i) X$$

But $ad(X_\alpha)$ is nilpotent $\Rightarrow \sum_{i=1}^{t+1} b_i = 0 \Rightarrow ad(X_\alpha)X = 0$. $\square$

6 Casimir Element

Let $(X_i)_{i=1}^n$ be a basis of $\mathfrak{g}$ and let $(X^i)_{i=1}^n$ be the dual basis w.r.t. the Killing form $(\cdot, \cdot)$. Then the Casimir element [1] of $\mathfrak{g}$ is the element of $\mathcal{U}(\mathfrak{g})$ defined by:

$$C = \sum_i X_i X^i$$

C is independent of the choice of basis. For suppose $(Y_j)_{j=1}^n$ is another basis, with dual basis $(Y^j)_{j=1}^n$. Then we have scalars a_j^i and b_r^j s.t. $Y_j = \sum_i a_j^i X_i$ and $Y^j = \sum_r b_r^j X^r$. Now

$$\delta_j^k = (Y_j, Y^k) = \sum_{i,r} a_j^i b_r^k (X_i, X^r) = \sum_i a_j^i b_i^k$$

$$\begin{aligned} \Rightarrow \sum_j Y_j Y^j &= \sum_j \Big(\sum_i a_j^i X_i \Big) \Big(\sum_r b_r^j X^r \Big) = \sum_{i,j,r} a_j^i b_r^j X_i X^r \\ &= \sum_{i,r} \Big(\sum_j a_j^i b_r^j \Big) X_i X^r = \sum_{i,r} \delta_r^i X_i X^r = \sum_i X_i X^i \end{aligned}$$

[1] The Casimir element is an incarnation of the Laplacian.

Now suppose $g \in Int(\mathfrak{g})$. The Killing form is preserved by inner automorphisms, so $(g \cdot X^i)$ is a dual basis to $(g \cdot X_i)$. Therefore C is invariant under g:

$$g \cdot C = \sum_i (g \cdot X_i)(g \cdot X^i) = C$$

by the previous result on independence of basis. Consequently, $C \in \mathcal{Z}(\mathfrak{g})$.

Example Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with the standard basis H, X, Y. Calculating the value of the Killing form[2] for different pairs from this basis, we find the only non-zero results are $(H, H) = 8$ and $(X, Y) = 4$. Therefore the dual basis is $H^* = H/8$, $X^* = Y/4$, and $Y^* = X/4$. And the Casimir element is given by

$$C = HH^* + XX^* + YY^* = \frac{H^2}{8} + \frac{XY}{4} + \frac{YX}{4}$$

We will also find it convenient to work with $\Omega = 1 + 8C = 1 + H^2 + 2(XY + YX)$.

References

[1] W. Fulton and J. Harris. *Representation Theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[2] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1972.

[3] N. Jacobson. *Lie Algebras*. Dover Publications,Inc., New York, 1979.

[2] The calculations are easier if we first observe that the Killing form (A,B) must be a constant times the form $trace(AB)$ since both are non-degenerate. The constant is easily calculated to be 4 by looking at $trace(H^2)$.