Representations of Real Reductive Groups - II

Amber Habib

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1 Introduction

In this lecture we work out the representation theory of $SL(2,\mathbb{R})$. The plan is to first look at the K-types and their possible multiplicities to get a list of irreducible $(\mathfrak{g}, K)$-modules, to realize these through explicit constructions, and to decide which can be made unitary. To preserve the flow of our narrative, some of the derivations have been postponed to Appendices.

Our main references are Taylor[2] and Varadarajan[3].

2 Structure of $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$, $\mathfrak{g}_o = \mathfrak{sl}(2,\mathbb{R})$. Name the standard basis of $\mathfrak{g}_o$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H,X] = 2X,\ [H,Y] = -2Y,\ [X,Y] = H$$

The Iwasawa decomposition consists of $K = (u_\theta)$, $A = (a_t)$, and $N = (n_s)$, where:

$$\begin{aligned} u_\theta &= \exp\theta(X-Y) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ a_t &= \exp tH = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \\ n_s &= \exp sX = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Consider an arbitrary element of $SL(2,\mathbb{R})$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then its Iwasawa decomposition $g = u_\theta a_t n_s$ is given by

$$e^{i\theta} = \frac{a - ic}{\sqrt{a^2+c^2}},\ e^t = \sqrt{a^2+c^2},\ s = \frac{ab+cd}{\sqrt{a^2+c^2}} \tag{1}$$

Since K is generated by $\exp\theta(X-Y)$, so that $\mathfrak{k}_o = \mathbb{R}\cdot(X-Y)$, we choose $\mathfrak{h} = \mathbb{C}\cdot(X-Y)$ as our Cartan subalgebra for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. This gives the standard basis for $\mathfrak{g}$:

$$H' = -i(X-Y),\ X' = (H+i(X+Y))/2,\ Y' = (H-i(X+Y))/2$$

$H \mapsto H'$, $X \mapsto X'$, $Y \mapsto Y'$ is an automorphism of $\mathfrak{g}$. Now a $\mathfrak{g}$-module V will be a $(\mathfrak{g},K)$-module if H' acts semisimply, with integral eigenvalues. Let us write $s(H';V)$ for the set of eigenvalues (weights) of H' in V. The centre $\mathcal{Z}$ of $\mathcal{U}(\mathfrak{g})$ is generated by

$$\Omega = (H+1)^2 + 4YX = (H'+1)^2 + 4Y'X'$$

(See Appendix A) For any $c \in \mathbb{C}$, write χ_c for the character that takes Ω to c.

3 Classification of Irreducible $(\mathfrak{g},K)$-modules

Lemma 3.1 *Let V be a $(\mathfrak{g},K)$-module, and $0 \neq v \in V(\xi)$. Let $M(v)$ be the cyclic module generated by v. Then $M(v)\cap V(\xi) = \mathbb{C}\cdot v$ iff v is an eigenvector for $Y'X'$.*

Proof. (only if) $Y'X'$ commutes with H', hence preserves $V(\xi)$.
(if) Let $Y'X'v = av$. Define

$$M = \mathbb{C}\cdot v + \sum_{n\geq 1}(\mathbb{C}\cdot X'^n v) + \sum_{n\geq 1}(\mathbb{C}\cdot Y'^n v)$$

We have the following relations between X' and Y':

$$X'Y'^n = Y'^n X' + nY'^{n-1}(H'-n+1) \qquad (2)$$
$$Y'X'^n = X'^n Y' - nX'^{n-1}(H'+n-1) \qquad (3)$$

These yield

$$\begin{aligned} X'Y'^n v &= (a+n(\xi-n+1))Y'^{n-1}v \\ Y'X'^n v &= (a-(n-1)(\xi+n))X'^{n-1}v \end{aligned}$$

It follows that M is $\mathfrak{g}$-stable and hence equals $M(v)$. Now the sums in M are all direct (the terms are from different K-isotypes), so the only intersection of M with $V(\xi)$ is $\mathbb{C}\cdot v$. $\square$

Corollary 3.2 *Let V be an irreducible $(\mathfrak{g},K)$-module. Then $V = M$ as defined above (for some ξ,v), with infinitesimal character χ_c, where $c = (\xi+1)^2+4a$. Also $s(H';V) \subset \xi + 2\mathbb{Z}$, with equality iff X' and Y' act injectively.*

Note that this implies that each non-zero $V(\xi)$ has dimension 1. We then say V is *multiplicity free*.

Lemma 3.3 *The converse to the above corollary holds. ie, if V is multiplicity free, $s(H'; V) \subset \xi + 2\mathbb{Z}$ for some ξ, and both X', Y' act injectively, then they act bijectively, $s(H'; V) = \xi + 2\mathbb{Z}$ and V is irreducible.*

Now we can draw a list of the possibilities. First we have the modules of *full orbit*, where $s(H'; V) = \xi + 2\mathbb{Z}$. These we label $P(c : E)$, where $c \in \mathbb{C}$, $E = \xi + 2\mathbb{Z}$, $\pm c^{1/2} - 1 \notin E$. The module is given a basis $(v_n)_{n\in\mathbb{Z}}$. The action is defined by first solving $c = (\xi + 1)^2 + 4a$ for a, and then defining:

$$\begin{aligned}
&H'v_n = (\xi + 2n)v_n \ (n \in \mathbb{Z}) \\
&X'v_n = v_{n+1}, Y'v_{n+1} = (a - n(\xi + n + 1))v_n \ (n \geq 0) \\
&Y'v_n = v_{n-1}, X'v_{n-1} = (a - (n-1)(\xi + n))v_n \ (n \leq 0)
\end{aligned}$$

That this is a well-defined $\mathfrak{g}$-module follows from the relations (2,3). It is independent of the choice of ξ within E. It has $s(H') = \xi + 2\mathbb{Z}$, inf. character χ_c, and is always irreducible.

Now let us look at the possibilities when at least one of X', Y', is not injective. This leads to three cases:

$D(\mu : +)$, $\mu \in \mathbb{C}$, has basis $(v_n)_{n\geq 0}$, with actions:

$$\begin{aligned}
&H'v_n = (\mu - 1 - 2n)v_n, Y'v_n = v_{n+1} \\
&X'v_n = n(\mu - n)v_{n-1}
\end{aligned}$$

This has $s(H') = \{\ldots, \mu - 3, \mu - 1\}$ and inf. character χ_{μ^2}. It is irreducible iff $\mu \neq 1, 2, \ldots$

$D(\mu : -)$, $\mu \in \mathbb{C}$, has basis $(v_n)_{n\geq 0}$, with actions:

$$\begin{aligned}
&H'v_n = (\mu + 1 + 2n)v_n, X'v_n = v_{n+1} \\
&Y'v_n = -n(\mu + n)v_{n-1}
\end{aligned}$$

This has $s(H') = \{\mu + 1, \mu + 3, \ldots\}$ and inf. character χ_{μ^2}. It is irreducible iff $\mu \neq -1, -2, \ldots$

$F(\mu)$, $\mu \in \mathbb{Z}_{\geq 0}$. When $\mu = 1, 2, \ldots$, $D(\mu : +)$ is reducible. Then $\sum_{r\geq 0} \mathbb{C} \cdot v_{\mu+r}$ is the only non-zero proper submodule and is isomorphic to $D(-\mu : +)$. Then set $F(\mu) = D(\mu, +)/D(-\mu : +)$. $F(\mu)$ has dimension μ, it is always irreducible, and its inf. character is χ_{μ^2}.

Theorem 3.4 *The irreducible $(\mathfrak{g}, K)$-modules are the $D(\mu : +)(\mu = 0, -1, \ldots)$, the $D(\mu : -)(\mu = 0, 1, \ldots)$, the $F(\mu)(\mu = 1, 2, \ldots)$, the $P(\mu^2 : 2\mathbb{Z})(\mu \neq$ odd integer), and $P(\mu^2 : 1 + 2\mathbb{Z})(\mu \neq$ even integer).*

4 Principal Series

Although we have constructed the irreducible $(\mathfrak{g}, K)$-modules, our task is not complete. We still need to relate them to the analytic or geometric constructions of G-modules. Questions of unitarity, in particular, are usually easier to settle from the latter perspective.

The *principal series* of G is obtained by inducing from one-dimensional representations of the Borel subgroup. These can be realized as follows. (See Appendix B)

For $\varepsilon = 0$ or 1, let $H^\varepsilon \subset L^2(K)$ be the Hilbert space of functions f such that $f(u_{-\theta}) = (-1)^\varepsilon f(u_\theta)$. Let $e_n(u_\theta) = e^{-in\theta}$. Then H^ε has Hilbert space basis $(e_{\varepsilon+2k})_{k\in\mathbb{Z}}$. The principal series representation $\pi_{\varepsilon,\lambda}$, $\lambda \in \mathbb{C}$, is defined by:

$$(\pi_{\varepsilon,\lambda}(x)f)(u_\theta) = e^{-(\lambda+1)t(x^{-1}u_\theta)}f(x^{-1}[u_\theta])$$

The exponential factor preserves unitarity. So $\pi_{\varepsilon,\lambda}$ is unitary when $\lambda \in i\mathbb{R}$, and then it is a member of the *unitary principal series*. Using the Iwasawa decomposition, we can calculate (Appendix C) the explicit actions of the Lie algebra elements.

Theorem 4.1 *The action of $\mathfrak{g}$ on the basis elements e_n in the representation $\pi_{\varepsilon,\lambda}$ is:*

$$\begin{aligned}
H' &: \quad e_n \mapsto ne_n, \\
X' &: \quad e_n \mapsto \frac{1}{2}(\lambda+n+1)e_{n+2}, \\
Y' &: \quad e_n \mapsto \frac{1}{2}(\lambda-n+1)e_{n-2} \\
\Omega &= \lambda^2
\end{aligned}$$

Theorem 4.2 *The actions described above lead to the following list:*
(1) If $\lambda \notin \mathbb{Z}$ then $\pi_{\varepsilon,\lambda}$ is irreducible and isomorphic to $P(\lambda^2 : \varepsilon + 2\mathbb{Z})$.
(2) If $\lambda \in \mathbb{Z}$ then $\pi_{\varepsilon,\lambda}$ is irreducible if λ and ε have the same parity, and then it is isomorphic to $P(\lambda^2 : \varepsilon + 2\mathbb{Z})$.
(3) Suppose $\lambda \in \mathbb{Z}$ and λ, ε have opposite parity. Then $X' : e_{-\lambda-1} \mapsto 0$ and $Y' : e_{\lambda+1} \mapsto 0$. So

$$\begin{aligned}
\langle \ldots, e_{-\lambda-3}, e_{-\lambda-1}\rangle &\cong D(-\lambda : +) \\
\langle e_{\lambda+1}, e_{\lambda+3}, \ldots\rangle &\cong D(\lambda : -)
\end{aligned}$$

If $\lambda > 0$, these modules are irreducible and linearly independent, and
$0 \to D(-\lambda : +) \oplus D(\lambda : -) \to \pi_{\varepsilon,\lambda} \to F(\lambda)$ is exact.
If $\lambda < 0$, $D(-\lambda : +) \cap D(\lambda : -) \cong F(-\lambda)$.
If $\lambda = 0$ (and $\varepsilon = 1$), then $\pi_{1,0} \cong D(0 : +) \oplus D(0 : -)$.

Note that $\pi_{1,0}$ is part of the unitary principal series but is not irreducible.

Corollary 4.3 *Every irreducible representation of G is a subrepresentation of some $\pi_{\varepsilon,\lambda}$.*

5 Unitarity

The decomposition into K-isotypes of a unitary representation is orthogonal. In our special situation, this means we have to scale the inner product on each one-dimensional K-type so that the elements H, X, Y of $\mathfrak{g}_o$ act as skew-Hermitian operators. We will need the following identities:

$$\begin{aligned} H &= X' + Y' \\ X &= \frac{i}{2}(H' - (X' - Y')) \\ Y &= -\frac{i}{2}(H' + (X' - Y')) \end{aligned}$$

Consider the representations $P(c : E)$, with basis (v_n) as described in Section 3. Then the desired equation $\langle Hv_n, v_m\rangle = -\langle v_n, Hv_m\rangle$ leads to the equality

$$(a - n(\xi + n + 1))||v_n||^2 = -||v_{n+1}||^2$$

and this is also the requirement for the actions of X and Y to be skew-Hermitian. So we can have a unitary structure for $P(c : E)$ iff

$$a - n(\xi + n + 1) < 0, \ \forall n \in \mathbb{Z}$$

We can choose $\xi = 0, -1$. For $\xi = 0$, the requirement becomes $n^2 - n - a > 0$. This gives $a < 0$ and $c = (\xi + 1)^2 + 4a < 1$. For $\xi = -1$, we get $a - n^2 < 0$, giving $a < 0$ and $c < 0$. Consequently, the irreducible unitary $P(c : E)$ can be listed as follows:

First Principal Series: $P(c : 2\mathbb{Z})$, with $c \in (-\infty, 0]$
Second Principal Series: $P(c : 1 + 2\mathbb{Z})$, with $c \in (-\infty, 0)$
Complementary Series: $P(c : 2\mathbb{Z})$, with $c \in (0, 1)$

Similar, but simpler, computations show that all the irreducible $D(\mu : \pm)$ modules are unitary. We label them as follows:

Holomorphic Discrete Series: $D(\mu : -)$, with $\mu = 1, 2, \ldots$
Anti-Holomorphic Discrete Series: $D(\mu : +)$, with $\mu = -1, -2, \ldots$
Mock Discrete Series: $D(0 : \pm)$

However, the non-trivial finite-dimensional $F(\mu)$ cannot be unitarized. The only normal subgroups of $SL(2, \mathbb{R})$ are $\{I\}$, $\{\pm I\}$ and $SL(2, \mathbb{R})$. Consequently, its only possible homomorphism into a $U(n)$, which is compact, is the trivial one. So our list is completed by the

Trivial Representation: $F(1)$

A $\mathcal{Z}(\mathfrak{sl}(2, \mathbb{C}))$

For a general reductive Lie algebra, let (E^i) be a basis, with dual basis (F_i) (wrt the Killing form). The Casimir element is then defined as $c = \sum_i E^i F_i \in \mathcal{U}(\mathfrak{g})$. It is independent of the basis and always belongs to $\mathcal{Z}$. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the dual basis to H, X, Y, is $H/8, Y/4$ and $X/4$ and $c = (1/8)(H^2+2XY+2YX) = (1/8)(H^2 + 2H + 4YX)$. So $\Omega = 8c + 1 \in \mathcal{Z}$.

Let us work out the Harish-Chandra homomorphism as described in Section 4 of Lecture I. The projection $p : \mathcal{Z} \to \mathcal{U}(\mathfrak{h})$ maps $\Omega \mapsto (H+1)^2$ and therefore $\Omega^n \mapsto (H+1)^{2n}$. $\delta = 1$ implies that $u(h) = h-1 \; \forall h \in \mathfrak{h}$ and so $u(H+1)^m = H^m$. Putting this together, we get the values of the Harish-Chandra homomorphism γ on the subalgebra generated by Ω: $\gamma(\Omega^n) = H^{2n}$.

Now we use the isomorphism part of Theorem I.4.2. The Weyl group consists of ± 1 and so the elements of $\mathcal{U}(\mathfrak{h})^{W(\triangle)}$ are those with even powers of H. These are exactly the terms that show up when γ acts on powers of Ω, so those powers must span $\mathcal{Z}$.

B Induced Representations

Let B be the *Borel subgroup* of upper-triangular matrices. Corresponding to the Iwasawa decomposition $G = KAN$, there is the *Levy-Langlands decomposition* $B = MAN$, where $M = \{\pm I\}$. Let χ be a quasi-character of B. Then $\chi = (\xi, \eta)$, where ξ is a character of M, and $\eta : A \to \mathbb{C}^\times$ is a quasicharacter of A. N acts trivially.

Let $X = G/B$, with measure μ obtained from the Haar measure on G. This measure is not G-invariant, but we have (as for all Lie groups):

$$d\mu_g/d\mu(x) = \rho_g(x) \in L^\infty_{loc}(X)$$

where $\mu_g(\mathcal{B}) = \mu(g^{-1} \cdot \mathcal{B})$.

Now we can describe π_χ, the representation of G induced from χ. The representation space is

$$H = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(gb) = \chi(b^{-1})f(g) \;\forall g \in G, b \in B \\ \int_X ||f([g])||^2 d\mu([g]) < \infty \end{array} \right\}$$

with inner product $\langle f_1, f_2 \rangle = \int_X \langle f_1([g]), f_2([g]) \rangle d\mu([g])$. The G action is

$$(x \cdot f)(g) = \rho_x(g)^{1/2} f(x^{-1}g)$$

The $\rho_x^{1/2}$ factor keeps the representation unitary when χ is unitary.

Note that $X \cong K/M$. We can now represent H as functions on K with an invariance property wrt M. Let $\varepsilon = 0, 1$ represent the two characters of M. Then

$$H \cong H^\varepsilon = \left\{ f \in L^2(K) | \; f(u_{-\theta}) = (-1)^\varepsilon f(u_\theta) \right\}$$

The identification of H with H^ε is that $f \in H^\varepsilon$ corresponds to a $g \in H$ defined by $g(kan) = \eta(a^{-1})f(k)$, and then $f = g|_K$. $K \cong G/A \cdot N$ gives an action of G on K and we denote this $x[u]$, $x \in G$, $u \in K$. These identifications give the G-action on H^ε as

$$(x \cdot f)(u) = \rho_x(u)\eta(a(x^{-1}u))^{-1} f(x^{-1}[u]) \tag{4}$$

where $\rho_x(u) = d(x^{-1}[u])/du$, and $a()$ is the projection from G onto A. The measures involved can be written explicitly (see [3, p 82-83]), and give $\rho_x(u) = \delta(a(x^{-1}u))^{-2}$ where

$$\delta \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = a_1$$

The quasi-character η has the form

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto a_1^\lambda$$

for some fixed $\lambda \in \mathbb{C}$. Substituting these values into (3), we get

$$(x \cdot f)(u) = e^{-(\lambda+1)t(x^{-1}u)} f(x^{-1}[u])$$

and we denote this action by $\pi_{\varepsilon,\lambda}$.

C The $\mathcal{U}(\mathfrak{g})$ action on H^ε

Let us work out the action of H on H^ε. The actions of X and Y can be similarly calculated, and they combine to give the actions of H', X', and Y'. Let $x_r = \exp(rH)$. The H-action is defined by

$$\begin{aligned} H \cdot e_n(u_\theta) &= \left.\frac{d}{dr}\right|_{r=0} (\exp(rH) \cdot e_n)(u_\theta) \\ &= \left.\frac{d}{dr}\right|_{r=0} \exp((\lambda+1)t(x_r^{-1}u_\theta)) e_n(x_r^{-1}[u_\theta]) \end{aligned}$$

$$x_r^{-1}u_\theta = \exp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} e^r\cos\theta & e^r\sin\theta \\ -e^{-r}\sin\theta & e^{-r}\cos\theta \end{pmatrix}$$

Substituting these in equation(1), and writing $x_r^{-1}u_\theta = u_\varphi a_t n_s$, we get

$$e^{i\varphi} = \frac{e^r\cos\theta + ie^{-r}\sin\theta}{\sqrt{e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta}}, \quad e^t = \sqrt{e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta}$$

Now we can calculate

$$\begin{aligned} &H \cdot e_n(u_\theta) \\ &= \left.\frac{d}{dr}\right|_{r=0} \left(e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta\right)^{-\frac{\lambda+1}{2}} \left(\frac{e^r\cos\theta + ie^{-r}\sin\theta}{\sqrt{e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta}}\right)^n \\ &= \left(\frac{\lambda+1+n}{2}\right) e_{n+2}(u_\theta) + \left(\frac{\lambda+1-n}{2}\right) e_{n-2}(u_\theta) \end{aligned}$$

References

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[2] M. E. Taylor. *Noncommutative Harmonic Analysis*, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 1986.

[3] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, volume 16 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1989.