Representations of Real Reductive Groups - X

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1 Introduction

In Lecture IX we had introduced Zuckerman's cohomological induction as an algebraic analogue of the complex analytic construction of the discrete series. However, while we had described the basic construction and some of the nice properties (like duality and the vanishing theoems), we had not shown that the representations so constructed are interesting. In this lecture we try to remedy that somewhat.

The book of Knapp and Vogan[7] is indubitably *the* reference for this subject. In it are five main results. The *duality theorem* connects the left exact Zuckerman functor Γ with the right exact Bernstein functor Π and leads to an invariant Hermitian form on the induced representations, thus making their unitarity possible. The *signature theorem* identifies subspaces of the inducing and induced representations where the Hermitian forms have the same signature. These results make no positivity assumption on the inducing representation.

Further progress does require appropriate positivity conditions. The *irreducibility theorem* then guarantees that irreducibles induce to irreducibles. Similary, the *unitarity theorem* then says that unitary representations lead to unitary representations. Finally, the *transfer theorem* allows comparison of cohomological induction with Mackey induction and the Langlands classification.

In this lecture, we will not prove any of these major theorems, but rather try to illustrate the themes through the example of the holomorphic discrete series of $SL(2,\mathbb{R})$. This is a somewhat trivial case, in that the derived functors are not called for, but it serves well as an easy first example of the algebraic approach. We note that the general algebraic construction of the holomorphic discrete series is due to Harish-Chandra[3, 4] who also realised them as holomorphic sections of vector bundles, thus giving an independent proof of unitarity.

2 Holomorphic Discrete Series of $SL(2,\mathbb{R})$

We have $G = SL(2,\mathbb{R})$ and $K = SO(2)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexified Lie algebras of G and K. Then $\mathfrak{h} = \mathfrak{k}$ is in fact a Cartan subalgebra of $\mathfrak{g}$. Recall from Lecture II the basis $\{H', X', Y'\}$ of $\mathfrak{g}$, with $\mathfrak{k} = \mathbb{C}H'$, and the relations $[H', X'] = 2X'$, $[H', Y'] = -2Y'$, and $[X', Y'] = H'$. Then $\triangle(\mathfrak{g},\mathfrak{k}) = \{\pm\alpha\}$, where $\alpha(H') = 2$. Choose $\triangle^+(\mathfrak{g}, \mathfrak{k}) = \{\alpha\}$. The half-sum of the positive roots, ρ, is given by $\rho(H') = 1$.

As usual, we have $\mathfrak{n} = \mathbb{C}X'$, $\mathfrak{n}^- = \mathbb{C}Y'$, and $\mathfrak{b} = \mathfrak{k} \oplus \mathfrak{n}$.

For $\lambda \in \mathfrak{k}^*$, let $\mathfrak{k}$ act on $\mathbb{C}_{\lambda-\rho} \cong \mathbb{C}$ with weight $\lambda - \rho$. Extend this to a $\mathfrak{b}$-action by letting $\mathfrak{n}$ act trivially. This $(\mathfrak{b}, \mathfrak{k})$-module is a $(\mathfrak{b}, K)$-module whenever $\mathfrak{k}$ acts with integral eigenvalues, i.e. $\lambda = n\rho$, $n \in \mathbb{Z}$. Suppose it is so. Define the *Verma module*

$$V(\lambda) = \mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(\mathbb{C}_{\lambda-\rho}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

The notation $\otimes_{\mathcal{U}(\mathfrak{b})}$ denotes the identification $ub \otimes v = u \otimes bv$, for $b \in \mathfrak{b}$. The actions are given by

$$\begin{aligned} X \cdot (u \otimes v) &= Xu \otimes v, \ X \in \mathfrak{g} \\ k \cdot (u \otimes v) &= \mathrm{Ad}(k)u \otimes kv, \ k \in K \end{aligned}$$

With these actions $V(\lambda)$ is a $(\mathfrak{g}, K)$-module.

We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$, and hence $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})$ as a left $\mathcal{U}(\mathfrak{n}^-)$ module and right $\mathcal{U}(\mathfrak{b})$ module. So we have an isomorphism of $\mathcal{U}(\mathfrak{n}^-)$ modules:

$$\begin{aligned} V(\lambda) &\cong (\mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho} \\ &\cong \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} (\mathcal{U}(\mathfrak{b}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}) \qquad (1) \\ &\cong \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho} \end{aligned}$$

This shows $V(\lambda)$ is a cyclic highest weight representation with cyclic highest weight vector $1 \otimes 1$ of weight $\lambda - \rho$.

Theorem 2.1 *$V(\lambda)$ is irreducible if $\lambda = n\rho$, $n \leq 0$.*

Proof. Let N be a non-zero submodule of $V(\lambda)$. Then it has a highest weight vector. From (1), $V(\lambda)$ has $\{v_i = Y'^i \otimes 1\}_{i=0}^{\infty}$ as a basis of weight vectors. In $\mathcal{U}(\mathfrak{g})$ we have the commutation relation

$$[X', Y'^i] = iY'^{i-1}(H' - (i-1))$$

Now $X'v_i = 0$ implies

$$\begin{aligned} 0 &= [X', Y'^i]1 \otimes 1 = iY'^{i-1}(H' - (i-1)) \otimes 1 = iY'^{i-1} \otimes (H' - (i-1)) \cdot 1 \\ &= i[(\lambda - \rho)H' - (i-1)]Y'^{i-1} \otimes 1 = i[(n-1)\rho(H') - (i-1)]Y'^{i-1} \otimes 1 \\ &= i[n-i]Y'^{i-1} \otimes 1 \end{aligned}$$

So for N to be a proper submodule, we need $0 < i = n$, but $n \leq 0$ implies this is not possible. $\square$

Comparing the K-types (from (1)) of $V(n\rho)$ with the classification in Lecture II, we see that $V(n\rho)$ is the holomorphic discrete series representation $D(n, +)$ when

$n \neq 0$, and the mock discrete series $D(0,+)$ when $n=0$. The anti-holomorphic discrete series is obtained by starting from the opposite parabolic.[1]

3 Unitarity of Discrete Series

In Lecture II, we had constructed an invariant metric on the discrete series essentially by elimination. With our new realisation of the discrete series we can illustrate a more general method.

Let us set some notation. Let $\bar{\mathfrak{b}}$ denote conjugation (w.r.t. $\mathfrak{g}_0$) applied to $\mathfrak{b}$. $Z \in \mathcal{C}(\mathfrak{k}, K)$, becomes a member of $\mathcal{C}(\mathfrak{b}, K)$ by letting $\mathfrak{n}$ act by zero, and we denote this by $Z_{\mathfrak{b}}$. Similarly, we have $Z_{\bar{\mathfrak{b}}} \in \mathcal{C}(\bar{\mathfrak{b}}, K)$. Conjugation maps $\mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$, so that $\bar{\mathfrak{n}}$ and $\bar{\mathfrak{b}}$ correspond to the opposite choice of positive roots.

Take the pair $(\mathfrak{b}, K)$, and let $Z \in \mathcal{C}(\mathfrak{b}, K)$. Let $\bar{Z}$ have underlying real vector space Z but with the opposite complex structure. Its algebraic dual $\bar{Z}^*$ is then the algebraic conjugate-dual of Z, i.e., the space of all conjugate-linear maps from Z into $\mathbb{C}$. The $(\mathfrak{b}, K)$ action on it is given by

$$\begin{aligned} (kf)z &= f(k^{-1}z), \;\; k \in K, f \in Z^* \\ (Xf)z &= -f(Xz), \;\; X \in \mathfrak{b}, f \in Z^* \end{aligned}$$

Let Z^c denote the subspace of K-finite maps of Z^*, and set $Z^h = \bar{Z}^c \cong \overline{Z^c}$. Then $Z^h \in \mathcal{C}(\bar{\mathfrak{b}}, K)$. The same definition can be carried out in $\mathcal{C}(\mathfrak{g}, K)$.

If Z has an invariant Hermitian form $\phi(\cdot,\cdot)$, then there is an induced $(\mathfrak{b}, K)$-map $\phi^h : Z \to (Z^h)_{\mathfrak{b}} \cong (Z_{\bar{\mathfrak{b}}})^h$ given by $(\phi^h(u))(z) = \phi(u,z)$. The ind functor takes ϕ^h to $\operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(\phi^h) : \operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}) \to \operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}((Z_{\bar{\mathfrak{b}}})^h)$. To get an invariant Hermitian form on $\operatorname{ind}(Z)$ however, we need a map: $\operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}) \to [\operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}})]^h$. Let us start by recalling the following definition:

$$\operatorname{pro}(Z) = \operatorname{pro}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z) = \operatorname{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), Z)_K$$

This is a $\mathcal{C}(\mathfrak{g}, K)$-module under the following actions:

$$\begin{aligned} (k\varphi)(u) &= k(\varphi(\operatorname{Ad}(k)^{-1}u), \;\; k \in K \\ (X\varphi)(u) &= \varphi(uX), \;\; X \in \mathfrak{g} \end{aligned}$$

We note without proof that ind and pro are exact functors.

Lemma 3.1 $[\operatorname{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z)]^c \cong \operatorname{pro}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z^c)$.

Proof. At the vector space level, we have the isomorphism

$$\imath : \operatorname{Hom}_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} Z, \mathbb{C}) \to \operatorname{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), \operatorname{Hom}_{\mathbb{C}}(Z, \mathbb{C}))$$

[1]In fact, in Lecture II we had used the opposite labelling of holomorphic and anti-holomorphic. The choice seems to depend on the author consulted.

given by $(\imath\varphi)(u)v = \varphi(u^t \otimes v)$. The transpose map on $\mathcal{U}(\mathfrak{g})$, $u \mapsto u^t$, is defined inductively by $X^t = -X$ for $X \in \mathfrak{g}$ and $(uv)^t = v^t u^t$ for $u, v \in \mathcal{U}(\mathfrak{g})$. The inverse map $\jmath$ is given by $(\jmath\psi)(u \otimes v) = \psi(u^t)v$. This respects the $(\mathfrak{g}, K)$ actions and so we have an isomorphism of the K-finite subspaces:

$$[\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z)]^c = \mathrm{Hom}_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} Z, \mathbb{C})_K \cong \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), \mathrm{Hom}_{\mathbb{C}}(Z, \mathbb{C}))_K$$

We complete the proof by showing that

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), \mathrm{Hom}_{\mathbb{C}}(Z, \mathbb{C}))_K = \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), \mathrm{Hom}_{\mathbb{C}}(Z, \mathbb{C})_K)_K = \mathrm{pro}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z^c)$$

Let φ lie in the left term, and $u \in \mathcal{U}(\mathfrak{g})$. Then $\langle K\varphi\rangle$ and $\langle Ku\rangle$ are finite dimensional. From the K action on the left side we have $k(\varphi(u)) = (k\varphi)(ku)$, implying that $\langle K(\varphi(u))\rangle$ is finite dimensional. □

Lemma 3.2 $\mathrm{ind}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(\overline{Z_{\mathfrak{b}}}) \cong \overline{\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}})}$.

Proof. This is a matter of unravelling the changes in complex structure. The isomorphism is given by $x \otimes z \leftrightarrow \bar{x} \otimes z$. □

The two lemmas immediately give the following theorem:

Theorem 3.3 (Easy Duality) $[\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}})]^h \cong \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}((Z_{\mathfrak{b}})^h)$. □

Let $e_Z : \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}((Z_{\mathfrak{b}})^h) \to [\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}})]^h$ be this isomorphism. Using φ^h now to denote the map: $Z_{\bar{\mathfrak{b}}} \to (Z_{\mathfrak{b}})^h$ induced by the form $\varphi(\cdot,\cdot)$, we have the map

$$\mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(\varphi^h) : \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(Z_{\bar{\mathfrak{b}}}) \to \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}((Z_{\mathfrak{b}})^h)$$

To complete the construction of the invariant form, we need a map:

$$\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}) \to \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(Z_{\bar{\mathfrak{b}}})$$

For a start[2], we have the following identification (See Appendix A for the proof):

$$\mathrm{Hom}_{\mathfrak{g},K}(\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}), \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(Z_{\bar{\mathfrak{b}}})) \cong \mathrm{Hom}_{\mathfrak{k},K}(Z, Z) \tag{2}$$

Let φ_Z be the (non-zero) element on the left that corresponds to 1 on the right. Then $\psi_Z^h = e_Z \circ \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(\varphi^h) \circ \varphi_Z$ is the required map that defines an invariant Hermitian form on $\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z)$. This is called the “Shapovalov form”. If the form φ on Z is non-degenerate, then φ^h is an isomorphism, and so is $\mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(\varphi^h)$. Since e_Z is also an isomorphism, and φ_Z is non-zero, in this case ψ_Z^h will be non-zero. In particular, when $\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}Z$ is irreducible, ψ_Z^h is injective

[2]This forms part of what is called “Hard Duality”.

and the induced form ψ_Z on $\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K} Z$ is non-degenerate. The question that remains is: When is it positive-definite?

From Appendix A we have the formula $\varphi_Z(x \otimes z)(y) = p(yx)z$, where $p : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{k})$ is the projection given by the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{k}) \oplus (\mathfrak{n}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n})$. We can now write an explicit formula for the new invariant form, using the notation $x^* = \bar{x}^t$ for $x \in \mathcal{U}(\mathfrak{g})$:

$$\psi_Z(x \otimes z, x' \otimes z') = \varphi(p(x'^* x)z, z') \tag{3}$$

For the special case $Z = \mathbb{C}_{\lambda-\rho}$ normalise $\varphi(1,1) = 1$ to get a form ψ_λ on $V(\lambda)$:

$$\psi_\lambda(x \otimes 1, x' \otimes 1) = (\lambda - \rho)(p(x'^* x))$$

We remark that $H'^* = H'$, $X'^* = -Y'$, and $Y'^* = -X'$. From this, we can calculate the form ψ_λ on the basis $v_i = Y'^i \otimes 1$. First, an inductive argument gives

$$p(X'^i Y'^i) = i!\, H'(H'-1)\cdots(H'-i+1)$$

From this we deduce

$$\psi_\lambda(v_i, v_i) = (-1)^i i!\,(n-1)(n-2)\cdots(n-i) \tag{4}$$

and this number is positive for $n \leq 0$. One can similarly see that $\psi_\lambda(v_i, v_j) = 0$ for $i \neq j$, and so we have a positive-definite form on $D(n,+)$ for $n \leq 0$.

It is interesting to note that this form is degenerate for $n > 0$ — exactly when $V(n\rho)$ is reducible. Its kernel is then a submodule ($\cong D(-n,+)$), and quotienting by it gives the irreducible finite-dimensional representation $F(n)$ of $SL(2,\mathbb{R})$. The weight vectors in $F(n)$ are $v_0, \ldots, v_{n-1}$; and we see from (4) that ψ has alternating signs on these — hence the $F(n)$ are not unitarisable.

4 Generalisations

In the general case we start with a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, such that $\mathfrak{l}$ is the complexification of the Lie algebra $\mathfrak{l}_0$ of a subgroup L of G. Given $Z \in \mathcal{C}(\mathfrak{l}, L \cap K)$, we let $\mathfrak{u}$ act trivially to make Z a $(\mathfrak{q}, L \cap K)$-module, and then induce to get a $(\mathfrak{g}, L \cap K)$ module:

$$\mathrm{ind}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(Z) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} Z$$

Since we want to end up in $\mathcal{C}(\mathfrak{g}, K)$ we need to apply the Zuckerman or Bernstein derived functors to this module. However, there is an important special case – when G/K is Hermitian symmetric – where the second stage is not needed. Today, we content ourselves with the generalisations to this special case.

There are two equivalent characterisations for the case of G/K being Hermitian symmetric. The first is that $\mathfrak{k}$ is the Levi factor of a maximal parabolic subalgebra $\mathfrak{q}$ whose nilradical $\mathfrak{u}$ is commutative. The other is that the center of K is S^1. Note that $SL(2,\mathbb{R})/SO(2)$ has these properties and we have used them in the previous sections.

Now let G/K be Hermitian symmetric and consider an irreducible $Z \in \mathcal{C}(\mathfrak{k}, K)$. Let $\mathfrak{u}$ act trivially, and look at

$$\mathrm{ind}_{\mathfrak{q},K}^{\mathfrak{g},K}(Z) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} Z$$

All the identifications and maps in Section 3 still exist and lead to the Shapovalov form ψ_Z on $\mathrm{ind}_{\mathfrak{q},K}^{\mathfrak{g},K}(Z)$, given as before by formula (3), with p referring to the projection $p : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{k})$ given by the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{k}) \oplus (\mathfrak{u}^{-}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{u})$. Jantzen[5, 6] has shown that the Shapovalov form is the unique invariant Hermitian form on the Verma modules and its kernel is the maximal submodule. Quotienting by the kernel therefore gives the irreducible module with the same highest weight.

The articles[1, 2] show that all the unitary irreducible highest weight modules in this situation fit a classification of the following pattern. They are parametrised by discrete points on certain lines in $\mathfrak{t}^*$, where $\mathfrak{t}_0$ is a common Cartan subalgebra for $\mathfrak{k}_0$ and $\mathfrak{g}_0$. On each line, parametrised by a real parameter z, the Verma modules can be represented as the same vector space with varying actions. The form then varies as a polynomial in z and one investigates the nature of the changes in its signature as z varies. The points $z < 0$ correspond to the holomorphic discrete series, and the corresponding Verma modules are irreducible. They stay irreducible upto some point $z = a \geq 0$. Irreducibility implies non-degeneracy of the form and hence the signature of the form can't change until $z = a$, and unitarisability is maintained. However, at this point the irreducible quotient of the Verma module does become unitarisable. Eventually, as we increase z the signature will consist of both $+$'s and $-$'s and we will lose unitarity. In between, however, there a few evenly distributed points where the signature consists of only $+$'s and 0's and at these points the irreducible quotient will be unitarisable. The classification gives a and the subsequent points of unitarity for all the lines in question.

A Frobenius Reciprocity

The usual Frobenius reciprocities are available in the context of $(\mathfrak{g}, K)$ modules:

Theorem A.1 *Let $Z \in \mathcal{C}(\mathfrak{b}, K)$ and $V \in \mathcal{C}(\mathfrak{g}, K)$. Then*

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{pro}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}})) &\cong \mathrm{Hom}_{\mathfrak{b},K}(V, Z_{\mathfrak{b}}) \\ \mathrm{Hom}_{\mathfrak{g},K}(\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}), V) &\cong \mathrm{Hom}_{\mathfrak{b},K}(Z_{\mathfrak{b}}, V) \end{aligned}$$

□

In the first case, if Φ and Φ' are corresponding elements in the identification, then they are related by $\Phi(v)x = \Phi'(x \cdot v)$. The relation in the second case is more subtle and we content ourselves with writing it down for a simple situation later, which will be all we need for this lecture. These results lead to a proof of identification (3) in Section 3:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g},K}(\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}), \mathrm{pro}_{\bar{\mathfrak{b}},K}^{\mathfrak{g},K}(Z_{\bar{\mathfrak{b}}})) &\cong \mathrm{Hom}_{\bar{\mathfrak{b}},K}(\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}), Z_{\bar{\mathfrak{b}}}) \\
&\cong \mathrm{Hom}_{\bar{\mathfrak{b}},K}(\mathrm{ind}_{\mathfrak{k},K}^{\bar{\mathfrak{b}},K}(Z), Z_{\bar{\mathfrak{b}}}) \\
&\cong \mathrm{Hom}_{\mathfrak{k},K}(Z, Z)
\end{aligned}$$

The last identification in this sequence is a variation of the second case in Theorem A.1. If ϕ and ϕ' are corresponding elements in it, they satisfy $\phi(\bar{b} \otimes z) = \phi'(\gamma(\bar{b})z)$, where $\gamma : \mathcal{U}(\bar{\mathfrak{b}}) \to \mathcal{U}(\mathfrak{k})$ is the projection given by the decomposition $\mathcal{U}(\bar{\mathfrak{b}}) = \mathcal{U}(\mathfrak{k}) \oplus \mathcal{U}(\bar{\mathfrak{b}})\mathfrak{n}$.

Starting with $1 \in \mathrm{Hom}_{\mathfrak{k},K}(Z, Z)$ we can trace our way back through these explicit isomorphisms to get the formula for ϕ_Z in Section 3. We first get $\phi_1 \in \mathrm{Hom}_{\bar{\mathfrak{b}},K}(\mathrm{ind}_{\mathfrak{k},K}^{\bar{\mathfrak{b}},K}(Z), Z_{\bar{\mathfrak{b}}})$, defined by $\phi_1(\bar{\mathfrak{b}} \otimes z) = \gamma(\bar{\mathfrak{b}})z$. Then we get $\phi_2 \in \mathrm{Hom}_{\bar{\mathfrak{b}},K}(\mathrm{ind}_{\mathfrak{b},K}^{\mathfrak{g},K}(Z_{\mathfrak{b}}), Z_{\bar{\mathfrak{b}}})$ given by $\phi_2(x \otimes z) = p(x)z$. Finally, we get $\phi_Z(x \otimes z)y = \phi_2(yx \otimes z) = p(yx)z$.

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