Representations of Real Reductive Groups - I

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1 Introduction

First, we give a very brief summary of the probable contents of this series of lectures, skipping definitions and all details. These will be filled in later, or not! Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}_o$. Then our broad aim is to study the unitary dual $\hat{G}$: the set of equivalence classes of irreducible unitary representations of G.

The first step is to transfer the problem to an algebraic setting. Let π be a representation of G in a Hilbert space H. Then there is an associated algebraic object, called a $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ denotes the complexification of $\mathfrak{g}_o$, and K is a maximal compact sub-group of G. The study of unitary G-modules reduces to the study of these $(\mathfrak{g}, K)$-modules.

Langlands (1974) classified the irreducible $(\mathfrak{g}, K)$-modules. So the question now takes the form — which of these are unitary?

The Langlands construction starts with a parabolic, so parabolic induction both fits in naturally and yields unitary representations. But it is known that it does not yield all of them. There is a refinement of this approach, due to Zuckerman, known as cohomological parabolic induction. At present, this seems to be the best bet for understanding the unitary dual.

In these talks, the first step is essentially taken for granted. We write out the definitions and list the theorems that allow the shift to the algebraic setting, but skip all proofs. Even so, this should occupy the rest of this lecture.

The second and third lectures will calculate the possibilities, quite explicitly, for the groups $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$.

Emboldened by these successes, we will tackle the Langlands classification, spending another lecture or two on it. Finally, perhaps, something will be said about cohomological induction.

The pattern for these talks roughly follows that in a survey article by Wallach[2]. The fleshing out of some details in the first lecture, and some reorganisation, was influenced by Varadarajan[1] and Wallach[3].

2 The Groups Concerned

Throughout G is a real Lie group. K is a maximal compact subgroup. Their (real) Lie algebras are $\mathfrak{g}_o$ and $\mathfrak{k}_o$. We will denote the complexifications of these Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$.

Definition 2.1 *Let $\mathfrak{g}$ be a Lie algebra with center $\mathfrak{z}(\mathfrak{g})$. Then $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.*

The definition of a real reductive Lie group G starts with the requirement that its Lie algebra be reductive. Additional hypotheses are added to ease the analysis on the group, and these may vary with context and author. The *Harish-Chandra Class $\mathcal{H}$*, defined below, suffices for studying semisimple Lie groups by reduction to their parabolic subgroups.

Definition 2.2 *A real Lie group G is of class $\mathcal{H}$ if:*
(1) $\mathfrak{g}_o$ is reductive.
(2) G has finitely many connected components.
(3) $Ad(G) \subset G_C$, the connected complex adjoint group of $\mathfrak{g}$.
(4) The analytic subgroup G_1 of G, corresponding to $\mathfrak{g}_1 = [\mathfrak{g}_o, \mathfrak{g}_o]$ has finite center.

Wallach[2] has a more involved definition, which has the virtue that the properties of the groups are put in plain view. Let G_C be a subgroup of $GL(n, \mathbb{C})$ defined as the common zeros of a finite set of complex polynomials on $M_n(\mathbb{C})$, and closed under complex conjugation. $G_R = G_C \cap GL(n, \mathbb{R})$ is the group of *real points*.

Definition 2.3 *G is a real reductive group if it is a finite cover of an open subgroup G_o of G_R.*

All the classical groups exist as reductive groups of type G_R. A general connected semisimple Lie group with finite center, on the other hand, is a finite cover of the identity component of $G_R = Aut(\mathfrak{g}_o)$.

A real reductive group is of *inner type* if $\mathrm{Ad}(G)$is a subgroup of $\mathrm{Int}(\mathfrak{g})$. With this additional hypothesis, Wallach's definition becomes the same as the "usual" one.

Theorem 2.4 *Let G be a real reductive group as in (2.3). Then it has an Iwasawa decomposition $G = KAN$, where K is the maximal compact subgroup of G, A is abelian, and N is nilpotent.*

3 Admissible Representations

We want, as always, to shift from the group action to the Lie algebra action. This requires the group action on the vectors to be smooth, but it is not *a priori* obvious that there any vectors on which the group acts smoothly. First, we show that the subspace H^∞ on which the group has a C^∞ action is, in fact, dense in H.

Definition 3.1 *Let $v \in H$. Then $v \in H^\infty$ if the map $G \to H$: $g \mapsto \pi(g)v$ is C^∞.*

Lemma 3.2 (Gårding) *Let $f \in C_c^\infty(G)$. Then $\pi(f)v = \int_G f(x)\pi(x)v dx$ is in H^∞ and the span of all such vectors is dense in H.*

Proof. $\pi(y)\pi(f)v = \int_G f(x)\pi(yx)v dx = \int_G f(y^{-1}x)\pi(x)v dx$ and one can differentiate wrt y under the integral sign. To show such vectors are dense, we take a sequence f_n such that:

1. $f_n \in C_c^\infty(G)$,
2. $\mathrm{supp}(f_n) \downarrow 1$,
3. $\int_G f_n dx = 1 \forall n$.

Then $||\pi(f_n)v - v|| \to 0$ as $n \to \infty$. □

The smoothness of the vector v is equivalent to the smoothness of all the functions $g \mapsto \langle \pi(g)v, w\rangle$ (with w fixed for each map). This shows that H^∞ is invariant under the G action. The G and $\mathfrak{g}$ actions are related by:

$$\pi(x)\pi(X)\pi(x)^{-1} = \pi(Ad(x)X)$$

and H^∞ is also $\mathfrak{g}$-invariant.

The problem of globalising a Lie algebra action to the group is akin to reconstructing a function from its Taylor Series. Therefore, we in fact need to consider the analytic vectors.

Definition 3.3 *A vector $v \in H$ is analytic under G if the map $g \mapsto \pi(g)v$ is analytic. H^ω denotes the space of G analytic vectors.*

The following result formalises the transition from $\mathfrak{g}$ to G:

Theorem 3.4 *Let $H' \subset H^\omega$ be $\mathfrak{g}$-stable. Then $Cl(H')$ is G-stable.*

Definition 3.5 *A vector $v \in H$ is K-finite if the span of its K-orbit is finite-dimensional. Let H_K be the space of analytic K-finite vectors.*

Theorem 3.6 (Harish-Chandra) *Suppose G is a connected Lie group with Iwasawa decomposition G=KAN. Then H_K is dense in H.*

Proof. First, we get some vectors that are analytic for the $S = AN$ action. For $x \in S$ define $\omega(x) = max(||\pi(x)||, ||\pi(x^{-1})||)$. Take an $f \in L^1(S, \omega dx)$ which is also analytic under the left regular representation of S. Now define $w = \pi(f)v = \int_S f(x)\pi(x)v dx$. This is analytic under the S action.

For any vector w as above, and now with f analytic on K, define $\pi(f)w = \int_K f(x)\pi(k)w\, dk$. These vectors are analytic for G and dense in H. □

H_K is not in general stable under G. However, we have:

Lemma 3.7 *H_K is invariant under the $\mathfrak{g}$ and K actions, and decomposes as an algebraic direct sum of the K-isotypes:*

$$H_K = \sum_{\xi \in \hat{K}} H_K(\xi)$$

This collection of results determines our basic definition:

Definition 3.8 *Let V be a complex vector space, acted on by $\mathfrak{g}$ and K (both representations are denoted π). V is a $(\mathfrak{g}, K)$-module if:*
1. *(compatibility) $\pi(Ad(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$ for all $X \in \mathfrak{g}$, $k \in K$.*
2. *(K-finiteness) $V = \sum_{\xi \in \hat{K}} V(\xi)$, as algebraic direct sum.*

An important subset of $(\mathfrak{g}, K)$-modules is the class of *admissible* $(\mathfrak{g}, K)$-modules. These are the ones for which each $V(\xi)$ is finite-dimensional. The same definition is used for admissible G-modules. It is obvious that the admissibility of H is equivalent to that of H_K.

Theorem 3.9 (Harish-Chandra) *If V is an irreducible $(\mathfrak{g}, K)$-module, then it is admissible. Further, it equals H_K for some irreducible representation of G on a Hilbert space H.*

In fact, there are bounds on $\dim(V(\xi))$. If V is an irreducible $(\mathfrak{g}, K)$-module, then there is a constant $c = c(\mathfrak{g}, \xi)$ such that

$$\dim(V(\xi)) \leq c \cdot \dim(\xi)^2$$

Now let $\mathcal{Q}$ = elements of $\mathcal{U}(\mathfrak{g})$ which commute with $\mathcal{U}(\mathfrak{k})$, and $\mathcal{A} = \mathcal{U}(\mathfrak{k}) \cdot \mathcal{Q}$. Then, for irreducible V, each $V(\xi)$ is an irreducible $\mathcal{A}$-module.

Definition 3.10 *A $(\mathfrak{g}, K)$-module V is unitary if $\exists$ positive definite inner product $\langle , \rangle$ on V so that:*
(1) $\langle kv, kw \rangle = \langle v, w \rangle$, $k \in K$,
(2) $\langle Xv, w \rangle = -\langle v, Xw \rangle$, $X \in \mathfrak{g}_o$.

Theorem 3.11 (Harish-Chandra) *Consider an admissible $(\mathfrak{g}, K)$-module V. Then $V = H_K$ for some admissible unitary representation H of G iff V is unitary.*

Theorem 3.12 (Harish-Chandra) *If (π_i, H^i), $i = 1, 2$ are unitary, admissible representations of G then π_1, π_2 are unitarily equivalent iff H_K^1 and H_K^2 are isomorphic as $(\mathfrak{g}, K)$-modules.*

4 Infinitesimal Characters

Definition 4.1 *$\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ is the centre of $\mathcal{U}(\mathfrak{g})$.*

Let V be an admissible $(\mathfrak{g}, K)$-module. Then each K-isotype $V(\xi)$ is also $\mathcal{Z}$-stable and $\mathcal{Z}|_{V(\xi)}$ can be put in upper-triangular form. For any homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$, define

$$V_\chi = \{v \in V : \ (z - \chi(z))^d v = 0, \ \forall z \in \mathcal{Z}, \ d = d(v)\}$$

Then V is an algebraic direct sum of the V_χ as $(\mathfrak{g}, K)$-modules. If V is irreducible, then we get $V = V_\chi$ for some χ. Now let $W_\chi = \{v \in V : \ (z - \chi(z))v = 0, \ \forall z \in \mathcal{Z}\}$. This is a non-zero $(\mathfrak{g}, K)$-module, hence $V = W_\chi$. In this case, χ is called the *infinitesimal character* of V.

To investigate possible values for this infinitesimal character, we need to list some more structure. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\triangle$ the corresponding root system, $\triangle^+$ a choice of positive roots, $\mathfrak{n}$ the sum of the positive root spaces, $\mathfrak{n}^-$ the sum of the negative root spaces, and $W(\triangle)$ the Weyl group. Let $\Lambda \in \mathfrak{h}^*$. Extend Λ to a homomorphism $\mathcal{U}(\mathfrak{h}) \to \mathbb{C}$. The Weyl group acts on $\mathfrak{h}^*$ by $s \cdot \Lambda = \Lambda \circ s^{-1}$.

The Poincaré-Birkhoff-Witt theorem implies that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus \{\mathfrak{n}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}\}$. Projecting $\mathcal{Z}$ into the first factor gives a homomorphism $p : \mathcal{Z} \to \mathcal{U}(\mathfrak{h})$. So we could get a character by taking $\Lambda \circ p$, but we will first do a small shift that pays dividend later. Define $u : \mathfrak{h} \to \mathcal{U}(\mathfrak{h})$ by $u(h) = h - \delta(h) \cdot 1$, where δ is half the sum of the positive roots, and extend this to a homomorphism $\mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$. Now define $\gamma = u \circ p$.

Theorem 4.2 (Harish-Chandra) *(1) γ is independent of $\triangle^+$.*
(2) γ is injective and an isomorphism onto $\mathcal{U}(\mathfrak{h})^{W(\triangle)}$.
(3) Suppose $\chi : \mathcal{Z} \to \mathbb{C}$ is a homomorphism s.t. $\chi(1) = 1$. Then $\exists \Lambda \in \mathfrak{h}^*$ s.t. $\chi = \chi_\Lambda = \Lambda \circ \gamma$.
(4) $\chi_\Lambda = \chi_{\Lambda'}$ iff $\Lambda' = s\Lambda$ for some $s \in W(\triangle)$.

This theorem gives us a handle on $\mathcal{Z}$ and the characters.

References

[1] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, volume 16 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1989.

[2] N. Wallach. Representations of reductive Lie groups. *Proceedings of Symposia in Pure Mathematics*, 33, part 1:71–86, 1979.

[3] N. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1988.