Chapter 3

Representations of $SL(2, \mathbb{R})$

In this chapter we describe the unitary dual of $SL(2, \mathbb{R})$. The plan is to first look at the K-types and their possible multiplicities to get a list of the irreducible $(\mathfrak{g}, K)$ modules, to decide which of these can be made unitary, and then to realize them through explicit constructions. These constructions allow us to study the decay rates of the matrix coefficients, and hence to obtain a description of the regular representation. In the last section, we describe the topology of the unitary dual.[1]

3.1 Structure of $SL(2, \mathbb{R})$

The object of our attention is the group $G = SL(2, \mathbb{R})$ of 2×2 real matrices with determinant equal to one. Its Lie algebra $\mathfrak{g}_o = \mathfrak{sl}(2, \mathbb{R})$ consists of the 2×2 real matrices with trace equal to zero. The exponential map from the Lie algebra into the Lie group is the usual exponentiation of matrices. The complexification of $\mathfrak{sl}(2, \mathbb{R})$ is the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ of 2×2 complex matrices with trace equal to zero.

Both $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{C})$ have the following standard basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The basis elements have the following bracket relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

[1]References for this chapter are Bargmann [6], Lang [93], Taylor [128], and Varadarajan [133].

Iwasawa Decomposition

The Iwasawa decomposition $G = KAN$ consists of $K = \{u_\theta\}$, $A = \{a_t\}$, and $N = \{n_s\}$, where:

$$\begin{aligned} u_\theta &= \exp\theta(X-Y) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \ \theta \in [0, 2\pi]; \\ a_t &= \exp tH = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \ t \in \mathbb{R}; \\ n_s &= \exp sX = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \ s \in \mathbb{R}. \end{aligned}$$

Thus we have $K \cong S^1$, $A \cong \mathbb{R}$ and $N \cong \mathbb{R}$. Consider an arbitrary element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$. Then its Iwasawa decomposition $g = u_\theta a_t n_s$ is given by

$$e^{i\theta} = \frac{a - ic}{\sqrt{a^2+c^2}}, \qquad e^t = \sqrt{a^2+c^2}, \qquad s = \frac{ab+cd}{\sqrt{a^2+c^2}}. \tag{3.1}$$

SU(1,1)

The group $SU(1,1)$ consists of the determinant one 2×2 complex matrices which preserve the form $|z_1|^2 - |z_2|^2$. It is conjugate, and hence isomorphic, to $SL(2,\mathbb{R})$:

$$\begin{aligned} SU(1,1) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\} = T \cdot SL(2,\mathbb{R}) \cdot T^{-1}, \\ \text{where } T &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \end{aligned}$$

A maximal compact subgroup is $T \cdot K \cdot T^{-1} = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$. There is the following *polar decomposition*:

$$x = e_\phi x_t e_\psi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \tag{3.2}$$

where $0 \le \phi, \psi \le 2\pi$ and $0 \le t < \infty$. If we expand the right hand side of (3.2), we get the expression

$$\begin{pmatrix} e^{i(\phi+\psi)} \cosh t & e^{i(\phi-\psi)} \sinh t \\ e^{i(\psi-\phi)} \sinh t & e^{-i(\phi+\psi)} \cosh t \end{pmatrix}$$

and the truth of the assertion is immediate. It is also immediate that the decomposition is not quite unique: the value of t is unique but ϕ and ψ are only fixed upto $\pm\pi$.

Theorem 3.1.1 *Let μ be a left Haar measure on $SU(1,1)$. It can be normalized such that for any $f \in C_c(SU(1,1))$:*

$$\int_{SU(1,1)} f(x)\, d\mu(x) = \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} f(e_\phi x_t e_\psi) \sinh(2t)\, d\phi\, dt\, d\psi.$$

Proof. We use the general formula for the Haar measure of $SL(n, \mathbb{R})$ provided in Example 1.0.8. With $n = 2$ and writing our members of $SL(2, \mathbb{R})$ as $\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$ the Haar measure becomes

$$|z|^{-1}\, dx\, dy\, dz.$$

The coordinates x, y, z are connected to the polar coordinates t, ϕ, ψ by

$$\begin{aligned} x &= \sinh(t)\cos(\phi - \psi) - \cosh(t)\sin(\phi + \psi) \\ y &= \sinh(t)\cos(\phi - \psi) + \cosh(t)\sin(\phi + \psi) \\ z &= \cosh(t)\cos(\phi + \psi) + \sinh(t)\sin(\phi - \psi) \end{aligned}$$

Now we merely have to calculate the Jacobian of the coordinate change from x, y, z to t, ϕ, ψ. This is a messy exercise when done by hand, but is easily accomplished by Mathematica. The determinant of the Jacobian is $z\sinh(2t)$ and hence we have our result. □

3.2 Classification of Irreducible $(\mathfrak{g}, K)$ modules

Due to Theorem 2.2.13 a first step towards obtaining the unitary dual of G is to classify the irreducible $(\mathfrak{g}, K)$ modules.

Since K is generated by $\exp\theta(X - Y)$, so that $\mathfrak{k}_o = \mathbb{R}\cdot(X - Y)$, we choose $\mathfrak{h} = \mathbb{C}\cdot(X - Y)$ as our Cartan subalgebra for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. This gives the following basis for $\mathfrak{g}$:

$$H' = -i(X - Y), \quad X' = \frac{1}{2}(H + i(X + Y)), \quad Y' = \frac{1}{2}(H - i(X + Y)),$$

where X' and Y' are chosen so that the bracket relations of Example 1.3.5 are again satisfied by the new basis. Thus the linear map given by $H \mapsto H'$, $X \mapsto X'$, $Y \mapsto Y'$, is an automorphism of $\mathfrak{g}$.

Now a $(\mathfrak{g}, K)$ module V is a $\mathfrak{g}$ module on which H' acts semisimply (since $\mathfrak{k}_o$ does), with integral eigenvalues (so that the $\mathfrak{k}_o$ action lifts to K). Let us write $s(V)$ for the set of (integral) eigenvalues (weights) of H' in V. Then we have the following decomposition into K-isotypes:

$$V = \sum_{\xi \in s(V)} V(\xi).$$

A non-zero vector $v \in V(\xi)$ is called a *highest weight vector of weight* ξ if $X'v = 0$. Similarly, it is called a *lowest weight vector of weight* ξ if $Y'v = 0$.

Recall from Corollary 1.4.4 that the centre $\mathcal{Z}$ of $\mathcal{U}(\mathfrak{g})$ is generated by

$$\Omega = (H + 1)^2 + 4YX = (H' + 1)^2 + 4Y'X'.$$

For any $c \in \mathbb{C}$, write χ_c for the infinitesimal character that takes Ω to c.

Lemma 3.2.1 *Let V be a $(\mathfrak{g}, K)$ module, and $0 \neq v \in V(\xi)$. Let $M(v)$ be the cyclic module generated by v. Then $M(v) \cap V(\xi) = \mathbb{C} \cdot v$ if and only if v is an eigenvector for $Y'X'$.*

Proof.
(Only if) $Y'X'$ commutes with H' in $\mathcal{U}(\mathfrak{g})$, hence preserves $V(\xi)$.

(If) Let $Y'X'v = av$. Define

$$M = \mathbb{C} \cdot v + \sum_{n \geq 1} (\mathbb{C} \cdot X'^n v) + \sum_{n \geq 1} (\mathbb{C} \cdot Y'^n v).$$

We have the following relations in $\mathcal{U}(\mathfrak{g})$ between X' and Y':

$$\begin{aligned} X'Y'^n &= Y'^n X' + nY'^{n-1}(H' - n + 1), \\ Y'X'^n &= X'^n Y' - nX'^{n-1}(H' + n - 1). \end{aligned}$$

These yield

$$X'Y'^n v = (a + n(\xi - n + 1))Y'^{n-1} v, \tag{3.3}$$
$$Y'X'^n v = (a - (n-1)(\xi + n))X'^{n-1} v. \tag{3.4}$$

It follows that M is $\mathfrak{g}$-stable and hence equals $M(v)$. Now the sums in M are all direct (the terms are from different K-isotypes), so the only intersection of M with $V(\xi)$ is $\mathbb{C} \cdot v$. □

Theorem 3.2.2 *Let V be an irreducible $(\mathfrak{g}, K)$ module. Then $V = M$ as defined above (for some ξ,v), with infinitesimal character χ_c, where $c = (\xi + 1)^2 + 4a$. Also $s(V) \subset \xi + 2\mathbb{Z}$, with equality iff X' and Y' act injectively. In any case, $s(V)$ will be an unbroken string: If the weight spaces $V(\xi + 2k)$ and $V(\xi + 2m)$ are both non-zero, with $k < m$, then each $V(\xi + 2l)$ is non-zero for $k < l < m$.*

Proof. According to Theorem 2.2.14, each $V(\xi)$ is finite-dimensional. [2] Pick any non-zero $V(\xi)$ — then $Y'X'$ acts on this finite dimensional space and therefore must have an eigenvector v in it. By the previous lemma it follows that $V = M$.

□

Note that this implies each non-zero $V(\xi)$ has dimension one. We therefore say V is *multiplicity free.*

Lemma 3.2.3 *The converse to the above corollary holds. i.e., if V is multiplicity free, $s(V) \subset \xi + 2\mathbb{Z}$ for some ξ, and both X', Y' act injectively, then they act bijectively, $s(V) = \xi + 2\mathbb{Z}$ and V is irreducible.* □

Now we can draw a list of the possibilities. The first case is when we have full spectrum: $s(V) = 2\mathbb{Z}$ or $1 + 2\mathbb{Z}$.

• $V = P(\varepsilon : c)$, where $c \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$, and $\pm c^{1/2} - 1 \notin \varepsilon + 2\mathbb{Z}$. These are the modules where $s(V) = \varepsilon + 2\mathbb{Z}$ and the infinitesimal character is χ_c.

[2] A direct proof of this for $SL(2,\mathbb{R})$ is in [127, Thm 5.1].

We choose a basis $(v_n)_{n\in\mathbb{Z}}$, where $v_0 \in V(\varepsilon)$, $X'v_n = v_{n+1}$ for $n \geq 0$, and $Y'v_n = v_{n-1}$ for $n \leq 0$. The action is defined by first solving $c = (\varepsilon+1)^2 + 4a$ for a, and then imposing the conditions (3.3) and (3.4):

$$\begin{array}{rcll} H'v_n &=& (\varepsilon + 2n)v_n & (n \in \mathbb{Z}), \\ Y'v_n &=& (a - (n-1)(\varepsilon + n))v_{n-1} & (n > 0), \\ X'v_n &=& (a - n(\varepsilon + n + 1))v_{n+1} & (n < 0). \end{array}$$

The condition on c ensures we have an irreducible module. For X' and Y' to be injective, we need $a - n(\varepsilon + n + 1) \neq 0\ \forall n \in \mathbb{Z}$. The requirement for this quadratic in n to have no integral zeroes is exactly that $\pm c^{1/2} - 1 \notin \varepsilon + 2\mathbb{Z}$.

Now let us look at the possibilities when at least one of X', Y', is not injective. This leads to three cases:

• $V = D(\mu : +)$, $\mu \in \mathbb{Z}$, has basis $(v_n)_{n\geq 0}$, with actions:

$$\begin{array}{rcl} H'v_n &=& (\mu - 1 - 2n)v_n, \\ X'v_n &=& n(\mu - n)v_{n-1}, \\ Y'v_n &=& v_{n+1}. \end{array}$$

The vector v_0 is a highest weight vector of weight $\mu - 1$. Further, $s(V) = \{\ldots, \mu-3, \mu-1\}$ and the infinitesimal character is χ_{μ^2}. $D(\mu : +)$ is irreducible if and only if $\mu \neq 1, 2, \ldots$

• $V = D(\mu : -)$, $\mu \in \mathbb{Z}$, has basis $(v_n)_{n\geq 0}$, with actions:

$$\begin{array}{rcl} H'v_n &=& (\mu + 1 + 2n)v_n, \\ X'v_n &=& v_{n+1}, \\ Y'v_n &=& -n(\mu + n)v_{n-1}. \end{array}$$

Here, v_0 is a lowest weight vector of weight $\mu + 1$. We have $s(H') = \{\mu + 1, \mu + 3, \ldots\}$ and infinitesimal character χ_{μ^2}. $D(\mu : -)$ is irreducible if and only if $\mu \neq -1, -2, \ldots$

• $V = F(n)$, $n \in \mathbb{Z}_{\geq 0}$. Suppose an irreducible $(\mathfrak{g}, K)$ module has both a highest weight and a lowest weight vector. Then it is finite dimensional. Recall that for each $n \in \mathbb{Z}$, $\mathfrak{sl}(2, \mathbb{C})$ has a unique finite dimensional module $F(n)$ of dimension n. A choice of basis elements of $F(n)$ is $\{v_i\}_{i=0}^{n-1}$, with actions

$$\begin{array}{rcl} H'v_i &=& (n - 1 - 2i)v_i, \\ X'v_i &=& i(n - i)v_{i-1} \text{ with } v_{-1} = 0, \\ Y'v_i &=& v_{i+1} \text{ with } v_n = 0. \end{array}$$

The set of weights of $F(n)$ is $\{-n+1, -n+3, \ldots, n-1\}$. Its infinitesimal character is χ_{n^2}.

Theorem 3.2.4 *The irreducible $(\mathfrak{g}, K)$ modules are the following:*

1. $P(0 : \lambda^2)$, $\lambda \neq$ odd integer,

2. $P(1 : \lambda^2)$, $\lambda \neq$ *even integer,*
3. $D(\mu : +)$, $\mu = 0, -1, -2, \dots,$
4. $D(\mu : -)$, $\mu = 0, 1, 2, \dots,$
5. $F(n)$, $n = 1, 2, \dots.$ □

3.3 Unitarizable Irreducible $(\mathfrak{g}, K)$ modules

The decomposition into K-isotypes of a unitary representation is orthogonal. In our special situation, this means we have to scale the inner product on each one-dimensional K-type so that the elements H, X, Y of $\mathfrak{g}_o$ act as skew-Hermitian operators. We will need the following identities:

$$\begin{aligned} H &= X' + Y', \\ X &= \frac{i}{2}(H' - (X' - Y')), \\ Y &= -\frac{i}{2}(H' + (X' - Y')). \end{aligned}$$

Consider the representations $P(\varepsilon : c)$, with basis (v_n) as described in §3.2. Then the desired equation $\langle Hv_n, v_m \rangle = -\langle v_n, Hv_m \rangle$ leads to the equality

$$(a - n(\varepsilon + n + 1))||v_n||^2 = -||v_{n+1}||^2 \tag{3.5}$$

and this is also the requirement for the actions of X and Y to be skew-Hermitian. So we can have a unitary structure for $P(\varepsilon : c)$ if and only if

$$a - n(\varepsilon + n + 1) < 0, \ \forall n \in \mathbb{Z}.$$

For $\varepsilon = 0$, the requirement becomes $n^2 + n - a > 0$. This gives $a < 0$ and $c = (\varepsilon + 1)^2 + 4a < 1$. For $\varepsilon = 1$, we get $a - n(n + 2) < 0$, giving $a < -1$ and $c < 0$. Consequently, the irreducible unitarizable $P(\varepsilon : c)$ can be listed as follows:

First Principal Series: $P(0 : c)$, with $c \in (-\infty, 0]$
Second Principal Series: $P(1 : c)$, with $c \in (-\infty, 0)$
Complementary Series: $P(0 : c)$, with $c \in (0, 1)$

Similar, but simpler, computations show that all the irreducible $D(\mu : \pm)$ are unitarizable. We label them as follows:

Holomorphic Discrete Series: $D(\mu : -)$, with $\mu = 1, 2, \dots$
Anti-holomorphic Discrete Series: $D(\mu : +)$, with $\mu = -1, -2, \dots$
Mock Discrete Series[3]: $D(0 : \pm)$

However, the non-trivial $F(\mu)$'s cannot be unitarized. So our list is completed by the

Trivial Representation: $F(1)$

We will justify the names of the different families in the subsequent sections of this chapter.

[3] Also called the *limit of discrete series*.

3.4 Principal Series

Let B be the *Borel subgroup* of upper-triangular matrices. Corresponding to the Iwasawa decomposition $G = KAN$, there is the *Levy-Langlands decomposition* $B = MAN$, where $M = \{\pm I\}$. Suppose ξ is a character of M, and η is a quasi-character of A.[4] Then $\chi = \xi \otimes \eta \otimes 1$ is a quasi-character of B, with N acting trivially. The quasi-character η has the form

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto a_1^{\lambda}$$

for some fixed $\lambda \in \mathbb{C}$.

We construct $\pi_\chi = \mathrm{Ind}_B^G(\chi)$, the representation of G induced from χ, following the procedure described in §2.5. We start with the space

$$C_c(G/B, \chi \otimes \triangle_{G/B}^{-1/2}) = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} 1.\ f \text{ is continuous and} \\ \quad \text{compactly supported mod } B. \\ 2.\ f(gb) = \triangle_{G/B}^{1/2}(b)\chi(b)^{-1} f(g) \\ \quad \forall g \in G,\ b \in B. \end{array} \right\}. \tag{3.6}$$

The group $G = SL(2,\mathbb{R})$ is reductive, hence unimodular, while $\triangle_B(man) = |\det \mathrm{Ad}_B(man)|$ can be calculated as follows. First $m = \pm I$ implies that $|\det \mathrm{Ad}_B(m)| = 1$. Next, $a = a_t \in A$ is of the form

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}.$$

Hence

$$|\det \mathrm{Ad}_B(a_t)| = \exp \left[\mathrm{Tr}\ \mathrm{ad}_B \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \right] = e^{2t}.$$

Proceeding similarly, we find that $|\det \mathrm{Ad}_B(n_s)| = 1$, and the invariance condition in (3.6) can be written explicitly as:

$$f(xma_tn) = e^{-(\lambda+1)t} \xi(m)^{-1} f(x).$$

Recall that when χ is unitary (which happens when λ is imaginary), we obtain an invariant positive definite form on $C_c(G/B, \chi \otimes \triangle_{G/B}^{-1/2})$ and on completing with respect to this we get a unitary representation of G, which we denote here by $(\pi_\chi, \mathcal{H}_\chi)$.

Compact picture

Note that $G/B \cong K/M$. We can now represent $\mathcal{H}_\chi$ as functions on K with an invariance property with respect to M. Let $\varepsilon = 0, 1$ represent the two characters of M. Then

$$\mathcal{H}^\varepsilon = \left\{ f \in L^2(K) |\ f(u_{-\theta}) = (-1)^\varepsilon f(u_\theta) \right\}. \tag{3.7}$$

[4] A quasi-character is a group homomorphism into $\mathbb{C}^\times$, while a character is also required to be unitary.

The identification of $\mathcal{H}_\chi$ with $\mathcal{H}^\varepsilon$ is that $f \in \mathcal{H}^\varepsilon$ corresponds to a $g \in \mathcal{H}_\chi$ defined by $g(kan) = \eta(a^{-1})f(k)$, and then $f = g|_K$. $K \cong G/A \cdot N$ gives an action of G on K and we denote this by $x[u]$, $x \in G$, $u \in K$. These identifications give the G action on $\mathcal{H}^\varepsilon$ as

$$(x \cdot f)(u) = e^{-(\lambda+1)t(x^{-1}u)} f(x^{-1}[u]) \tag{3.8}$$

This "compact picture" makes sense even when χ is not unitary (i.e., λ is an arbitrary complex number) and then it gives us a non-unitary representation of G, which we again denote by π_χ or $\pi_{\varepsilon,\lambda}$.

Let $e_n(u_\theta) = e^{-in\theta}$. Then $\mathcal{H}^\varepsilon$ has Hilbert space basis $(e_{\varepsilon+2k})_{k\in\mathbb{Z}}$. (We have the obvious identification of u_θ with $e^{-i\theta}$, so that e_n is really the function $z \mapsto z^n$.) Using the Iwasawa decomposition, we can calculate the explicit actions of the Lie algebra elements.

Theorem 3.4.1 *The action of $\mathfrak{g}$ on the basis elements e_n in the representation $\pi_{\varepsilon,\lambda}$ is:*

$$\begin{aligned}
H' &: e_n \mapsto n e_n,\\
X' &: e_n \mapsto \frac{1}{2}(\lambda+n+1)e_{n+2},\\
Y' &: e_n \mapsto \frac{1}{2}(\lambda-n+1)e_{n-2},\\
\Omega &= \lambda^2.
\end{aligned}$$

Proof. Let us work out the action of H on $\mathcal{H}^\varepsilon$. Let $x_r = \exp(rH)$. The H-action is defined by

$$\begin{aligned}
H \cdot e_n(u_\theta) &= \left.\frac{d}{dr}\right|_{r=0} (x_r \cdot e_n)(u_\theta)\\
&= \left.\frac{d}{dr}\right|_{r=0} e^{(\lambda+1)t(x_r^{-1}u_\theta)} e_n(x_r^{-1}[u_\theta]).
\end{aligned}$$

We have

$$x_r^{-1}u_\theta = \exp\begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} e^{-r}\cos\theta & e^{-r}\sin\theta \\ -e^r\sin\theta & e^r\cos\theta \end{pmatrix}.$$

Substituting this in (3.1), and writing $x_r^{-1}u_\theta = u_\varphi a_t n_s$, we get

$$e^{i\varphi} = \frac{e^{-r}\cos\theta + ie^r\sin\theta}{\sqrt{e^{-2r}\cos^2\theta + e^{2r}\sin^2\theta}}, \qquad e^t = \sqrt{e^{-2r}\cos^2\theta + e^{2r}\sin^2\theta}.$$

Now we can calculate

$$\begin{aligned}
&H \cdot e_n(u_\theta)\\
&= \left.\frac{d}{dr}\right|_{r=0} \left(e^{-2r}\cos^2\theta + e^{2r}\sin^2\theta\right)^{-\frac{1}{2}(\lambda+1)} \left(\frac{e^{-r}\cos\theta + ie^r\sin\theta}{\sqrt{e^{-2r}\cos^2\theta + e^{2r}\sin^2\theta}}\right)^{-n}\\
&= \left(\frac{\lambda+1+n}{2}\right) e_{n+2}(u_\theta) + \left(\frac{\lambda+1-n}{2}\right) e_{n-2}(u_\theta).
\end{aligned}$$

The actions of X and Y can be similarly calculated, and they combine to give the actions of H', X', and Y'. $\square$

Theorem 3.4.2 *The actions described above lead to the following list:*

1. If $\lambda \notin \mathbb{Z}$ then $\pi_{\varepsilon,\lambda}$ is irreducible and isomorphic to $P(\varepsilon : \lambda^2)$.

2. If $\lambda \in \mathbb{Z}$ then $\pi_{\varepsilon,\lambda}$ is irreducible if λ and ε have the same parity, and then it is isomorphic to $P(\varepsilon : \lambda^2)$.

3. Suppose $\lambda \in \mathbb{Z}$ and λ, ε have opposite parity. Then $X' : e_{-\lambda-1} \mapsto 0$ and $Y' : e_{\lambda+1} \mapsto 0$. So

$$\begin{aligned} \langle \ldots, e_{-\lambda-3}, e_{-\lambda-1} \rangle &\cong D(-\lambda : +), \\ \langle e_{\lambda+1}, e_{\lambda+3}, \ldots \rangle &\cong D(\lambda : -). \end{aligned}$$

If $\lambda > 0$, these modules are irreducible and linearly independent, and $0 \to D(-\lambda : +) \oplus D(\lambda : -) \to \pi_{\varepsilon,\lambda} \to F(\lambda)$ is exact.
If $\lambda < 0$, $D(-\lambda : +) \cap D(\lambda : -) \cong F(-\lambda)$.
If $\lambda = 0$ (and $\varepsilon = 1$), then $\pi_{1,0} \cong D(0 : +) \oplus D(0 : -)$. □

Thus the irreducible induced representations $\pi_{\varepsilon,\lambda}$ correspond exactly to the representations $P(\varepsilon, \lambda^2)$ in our earlier list of irreducible $(\mathfrak{g}, K)$ modules, and the irreducible unitary ones ($\lambda \in i\mathbb{R}$) correspond to the first and second principal series from that list. Therefore a general $\pi_{\varepsilon,\lambda}$ is called a *principal series* representation, and a unitary one is said to be a member of the *unitary principal series*. Note that $\pi_{1,0}$ is part of the unitary principal series but is not irreducible.

Corollary 3.4.3 *Every irreducible representation of G is infinitesimally equivalent to a subrepresentation of some principal series representation.* □

3.5 Discrete Series

The construction of the principal series involved an induction starting from the diagonal subgroup of $SL(2, \mathbb{R})$. The importance of this group was due to its complexified Lie algebra being a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. Another subgroup of G with this property is the rotation group K, and so we hope to construct further representations by an induction starting from K. However, K is fundamentally different as it is not contained in a Borel subgroup. (That is to say, it is not contained in a subgroup conjugate to B.) A way around this is to induce directly from K, but using complex analysis instead of real analysis. A hint in this direction is that in the compact picture, the two discrete series are embedded either in the holomorphic or in the anti-holomorphic part of $\mathcal{H}^\varepsilon$.

It is convenient to work with $SU(1,1)$. So in the rest of this section we use the notation $G = SU(1,1)$ and $K = \{u_\theta = \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) : \theta \in [0, 2\pi]\}$. G acts on $\mathbb{C}$ by fractional linear transformations:

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}.$$

This gives a transitive action of G on the open unit disc D such that the isotropy group of 0 is K. Hence G/K is isomorphic to D as a manifold and can be given the corresponding complex structure. Let ρ_n be the character of K defined by

$$\rho_n(u_\theta) = e^{-i(n+1)\theta}.$$

Since G and K are unimodular, G/K has a positive G invariant measure $d\lambda$. Then the unitary representation $\mathrm{Ind}_K^G(\rho_n)$ has the representation space

$$V_n = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} 1.\ f \text{ is measurable.} \\ 2.\ f(gk) = \rho_n(k)^{-1} f(g)\ \forall g \in G,\ k \in K. \\ 3.\ ||f||^2 = \int_{G/K} |f(g)|^2 d\lambda(\bar{g}) < \infty. \end{array} \right\},$$

with G acting by left multiplication.

Theorem 3.5.1 *For $z \in \mathbb{C}$ write $z = x + iy$, $x, y \in \mathbb{R}$. Then*

$$d\lambda = (1 - |z|^2)^{-2} dx\, dy$$

is a G invariant measure on D.

Proof. Let us write $z' = x' + iy' = g \cdot z$, where $g = \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$. The Jacobian of the transformation is given by

$$\begin{aligned} \frac{\partial(x', y')}{\partial(x, y)} &= \begin{vmatrix} \partial x'/\partial x & \partial y'/\partial x \\ \partial x'/\partial y & \partial y'/\partial y \end{vmatrix} \\ &= \left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2 \\ &= \left|\frac{dz'}{dz}\right|^2 = |\bar{\beta} z + \bar{\alpha}|^{-4}. \end{aligned}$$

Now one can easily calculate that

$$(1 - |z'|^2)^{-2}\, dx'\, dy' = (1 - |z'|^2)^{-2} |\bar{\beta} z + \bar{\alpha}|^{-4}\, dx\, dy = (1 - |z|^2)^{-2}\, dx\, dy.$$

□

The unitary representations V_n, for $n \geq 1$, have another realization which we now give. For $n \geq 1$, consider the representation whose space T_n consists of the measurable functions ϕ in the unit disc for which $||\phi||^2 < \infty$, where

$$||\phi||^2 = \int_{|z|<1} |\phi(z)|^2 (1 - |z|^2)^{n-1} dx\, dy,$$

and the G action is given by

$$(g \cdot \phi)(z) = (-\bar{\beta} z + \alpha)^{-(n+1)} \phi\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}\right).$$

Theorem 3.5.2 *The representations V_n and T_n $(n \geq 1)$ are unitarily isomorphic.*

Proof. For $\phi \in T_n$, and $g = \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$, define

$$A\phi(g) = \rho_n^c(\bar{\alpha}) \phi(g0)$$

where $\rho_n^c : \mathbb{C} \to \mathbb{C}$ is the map $z \mapsto z^{-(n+1)}$. One checks by direct calculations that $A\phi \in V_n$ and $||A\phi|| = ||\phi||$. Further, let $f \in V_n$. Define

$$f_1(g) = \rho_n^c(\bar{\alpha})^{-1} f(g).$$

Then $f(gk) = f(g)\ \forall k \in K$, and so f_1 induces a function ϕ on G/K. We directly check again that $\phi \in T_n$ and $A\phi = f$. Therefore $A : T_n \to V_n$ is a surjective isometry. It is also easy to see that it intertwines the G actions on V_n and T_n.

□

Let $\mathcal{H}_n$ be the subspace consisting of the holomorphic functions in T_n. Then $\mathcal{H}_n$ is a closed G invariant subspace of T_n and hence a unitary representation of G. (See Sugiura [127] for the details of this claim.) We denote the action of G on $\mathcal{H}_n$ by $\mathcal{D}_n$. In particular, the K-action is given by

$$(\mathcal{D}_n(u_\theta)f)(z) = e^{-i(n+1)\theta} f(e^{-2i\theta}z).$$

So the subspace of polynomials in $\mathcal{H}_n$ is the underlying $(\mathfrak{g}, K)$ module. The K-action on the basis $\{z^k\}_{k=0}^\infty$ is

$$\mathcal{D}_n(u_\theta)(z^k) = e^{-(n+1+2k)i\theta} z^k.$$

From here we can calculate that $TH'T^{-1}$, where $T = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$, acts on z^k with eigenvalue $n+1+2k$. Therefore the underlying $(\mathfrak{g}, K)$ module of $\mathcal{D}_n$ is $D(n : -)$, the holomorphic discrete series. The anti-holomorphic discrete series can be obtained by starting with the opposite choice of complex structure on G/K.

Theorem 3.5.3 *The representations $D(-n, +)$ and $D(n, -)$ are square integrable for $n > 0$.*

Proof. We show that each member has a square summable matrix coefficient and hence is a direct summand of $L^2(G)$. It suffices to do this for $D(n, -)$.

Consider $\mathcal{D}_n$, $n > 0$. The constant function 1 is the lowest weight vector, with weight $n + 1$. The corresponding matrix coefficient is

$$\begin{aligned} \varphi(x) &= \langle \mathcal{D}_n(x)1, 1\rangle \\ &= \int_{|z|<1} (-\bar{\beta}z + \alpha)^{-(n+1)}(1 - |z|^2)^{n-1} dx\, dy. \end{aligned}$$

From the unitarity of the action, we deduce that φ satisfies the following transformation law:

$$\varphi(e_\phi x e_\psi) = e^{-i(n+1)(\phi+\psi)}\varphi(x), \tag{3.9}$$

where the notation is as in (3.2). Now,

$$\begin{aligned} \varphi(x_t) &= \int_{|z|<1} (\sinh(t)z + \cosh(t))^{-(n+1)}(1 - |z|^2)^{n-1} dx\, dy \\ &= \int_0^1 \int_0^{2\pi} (\sinh(t)re^{i\theta} + \cosh(t))^{-(n+1)}(1 - r^2)^{n-1} r\, d\theta\, dr \\ &= \cosh(t)^{-(n+1)} \int_0^1 \Big(\int_0^{2\pi} (1 + \tanh(t)re^{i\theta})^{-(n+1)} d\theta\Big)(1 - r^2)^{n-1} r\, dr \\ &= 2\pi \cosh(t)^{-(n+1)} \int_0^1 (1 - r^2)^{n-1} r\, dr \\ &= \frac{\pi}{n} \cosh(t)^{-(n+1)}. \end{aligned}$$

Combining this result with (3.6) and Lemma 3.1.1 we find

$$\begin{aligned}\int_G |\varphi(x)|^2 d\mu(x) &= \frac{\pi^2}{n^2}\int_0^\infty \cosh(t)^{-2(n+1)}\sinh(2t)\,dt \\ &= \frac{2\pi^2}{n^2}\int_0^\infty \cosh(t)^{-(2n+1)}\sinh(t)\,dt \\ &= \frac{2\pi^2}{n^2}\int_1^\infty u^{-(2n+1)}du \;=\; \frac{\pi^2}{n^3} \;<\; \infty.\end{aligned}$$

□

The calculations that we carried out for one matrix coefficient can also be attempted for the rest. If we write a basis element as $f_k = z^k$, then the matrix coefficients of $\mathcal{D}_n$, $n > 0$, are given by

$$\begin{aligned}\varphi_{k,l}(x_t) &= \langle \mathcal{D}_n(x_t) f_k, f_l\rangle \\ = \int_{|z|<1} &((\sinh t)z + \cosh t)^{-(k+n+1)}((\cosh t)z + \sinh t)^k \bar{z}^l (1-|z|^2)^{n-1} dx\,dy.\end{aligned}$$

Although the integral is complicated, we can readily read off the asymptotic dependence of $\varphi_{k,l}(a_t)$ on t. It varies as $e^{-t(n+1)}$. Thus we also see directly that all the matrix coefficients are square-integrable. For $n \geq 2$, all the matrix coefficients $\varphi_{k,l}$ are in fact seen to be in $L^1(G) \cap L^2(G)$. Thus the representations $D(\mp n, \pm)$ are integrable for $n \geq 2$.

From Theorem 2.4.3 we know that the discrete series obeys the Schur orthogonality relations, with the formal degree playing the role of dimension. We note here that the truth of these relations was first observed by Bargmann for $SL(2,\mathbb{R})$ and then proved in general by Godement.

Theorem 3.5.4 *The discrete series representations $D(n,-)$ and $D(-n,+)$, $n \geq 1$, have formal degree n.*

Proof. As usual we write the proof only for $D(n,-)$. Consider its realisation in §3.5. Let $\varphi(x)$ be the matrix coefficient defined by $\langle x \cdot 1, 1\rangle$. In the proof of Theorem 3.5.3 we have calculated

$$\int_G |\varphi(x)|^2 d\mu(x) = \frac{\pi^2}{n^3}.$$

On the other hand, $||1||^2 = \pi/n$. Therefore the formal degree d satisfies

$$\frac{\pi^2}{n^3} = d^{-1}\left(\frac{\pi}{n}\right)^2,$$

and hence $d = n$. □

3.6 Mock Discrete Series

The above construction of the discrete series does not include the mock discrete series, for the space $\mathcal{H}_0$ is not defined. We give a construction of the mock discrete series through another route, and then show that this is a "limit" of the discrete series in a natural way. Let $\mathcal{D}_0$ be the representation whose space consists of the holomorphic functions f in the unit disc for which $||f||^2 < \infty$, where:

$$||f||^2 = \frac{1}{2\pi} \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

This is the well-known Hardy space H^2. The monomials z^k form a complete orthonormal set in it. Further, if $f = \sum a_n z^n \in H^2$, then

$$M(f : r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum |a_n|^2 r^{2n}. \tag{3.10}$$

The unitary action of G on H^2 is given by

$$(g \cdot f)(z) = (-\bar{\beta} z + \alpha)^{-1} f\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}\right).$$

Its underlying $(\mathfrak{g}, K)$ module is the mock discrete series $D(0 : -)$. Again, the opposite choice of complex structure on D would give $D(0 : +)$.

Theorem 3.6.1 *The mock discrete series representations $D(0 : \pm)$ are not square integrable. However, the matrix coefficient $\varphi(x) = \langle x \cdot 1, 1 \rangle$ is in $L^p(G)$ for every $p > 2$.*

Proof. We write the proof for $D(0 : -)$. Then the action of $g \in G$ on the function 1 is $(g \cdot 1)(z) = (-\bar{\beta} z + \alpha)^{-1}$. In particular,

$$\begin{aligned}
(x_t \cdot 1)(z) &= (-\sinh(t) z + \cosh(t))^{-1} \\
&= \cosh(t)^{-1} (1 - \tanh(t) z)^{-1} \\
&= \cosh(t)^{-1} \sum_{s=0}^{\infty} (\tanh(t) z)^s.
\end{aligned}$$

And

$$\varphi(x_t) = \langle x_t \cdot 1, 1 \rangle = \cosh(t)^{-1}.$$

Therefore $\varphi \in L^p(G)$ if and only if

$$\int_0^{\infty} (e^{-t})^p e^{2t} dt < \infty.$$

This happens precisely when $p > 2$. □

Consider again the realization of the discrete series in the spaces $\mathcal{H}_n$, $n \ge 1$. In fact the definition of these spaces makes sense when the parameter is allowed to vary over all of $(0, \infty)$. We now show that the mock discrete series $\mathcal{D}_0$ can be realized in a natural way as $\lim_{t \downarrow 0} \mathcal{H}_t$. First, we scale the original norm on $\mathcal{H}_t$:

$$||f||_t^2 = t \int_D |f(z)|^2 (1 - |z|^2)^{t-1} dx\, dy.$$

Then $\mathcal{H}_t$, $t > 0$, has a complete orthonormal set $\{\Psi_t^n\}_{n=0}^{\infty}$ defined by

$$\Psi_t^n(z) = \left[\frac{\Gamma(t + n + 1)}{n!\, \Gamma(t + 1)}\right]^{1/2} z^n.$$

Theorem 3.6.2 *We have the following equality:*

$$H^2 = \{f \in \cap_{t>0}\mathcal{H}_t : \lim_{t\downarrow 0} ||f||_t^2 \text{ exists and is finite.}\}.$$

Further, if $f \in H^2$, then $||f||^2 = \lim_{t\downarrow 0} ||f||_t^2$.

Proof. Let $f = \sum_{n=0}^{\infty} a_n z^n \in H^2$. Then

$$\begin{aligned} ||f||_t^2 &= \sum_{n=0}^{\infty} |\langle f, \Psi_t^n\rangle|^2 \\ &= \sum_{n=0}^{\infty} |a_n|^2 \left[\frac{n!\,\Gamma(t+1)}{\Gamma(t+n+1)}\right] \\ &\leq \sum_{n=0}^{\infty} |a_n|^2 = ||f||^2. \end{aligned}$$

It follows that $H^2 \subset \cap_{t>0}\mathcal{H}_t$. The series for $||f||_t^2$ converge absolutely and uniformly with respect to t, and hence

$$\lim_{t\downarrow 0} ||f||_t^2 = \sum_{n=0}^{\infty} |a_n|^2 \lim_{t\downarrow 0} \left[\frac{n!\,\Gamma(t+1)}{\Gamma(t+n+1)}\right] = \sum_{n=0}^{\infty} |a_n|^2 = ||f||^2.$$

For the reverse inclusion, let $f \in \cap_{t>0}\mathcal{H}_t$ such that $\lim_{t\downarrow 0} ||f||_t^2$ exists and is finite. Then

$$\begin{aligned} ||f||_t^2 &= \sum_{n=0}^{\infty} |a_n|^2 \left[\frac{n!\,\Gamma(t+1)}{\Gamma(t+n+1)}\right] \\ &\geq \sum_{n=0}^{\infty} |a_n|^2 (1+t)^{-n} \\ &= M(f:r), \end{aligned}$$

where $r = (1+t)^{-1/2}$ varies over all of $(0,1)$, and $r \to 1$ as $t \downarrow 0$. Therefore $||f||^2 = \sup_{0\leq r<1} M(f:r)$ exists and is finite. □

3.7 Matrix Coefficients of Principal Series

Let us find an explicit expression for the compact picture of the principal series of $SL(2,\mathbb{R})$. As in the previous section, we choose to work with $G = SU(1,1)$. It has an Iwasawa decomposition $G = KAN$: Every $x \in G$ decomposes as $x = kan$, where

$$\begin{aligned} k &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \exp\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}, \\ a &= \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} = \exp\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}, \\ n &= \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix} = \exp\begin{pmatrix} i & -i \\ i & -i \end{pmatrix}. \end{aligned}$$

If $x = \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$, then the decomposition is given by[5]

$$e^{i\theta} = \frac{\alpha+\beta}{|\alpha+\beta|}, \quad e^{s} = |\alpha+\beta|, \quad 2it = \frac{\alpha\bar{\beta}-\beta\bar{\alpha}}{|\alpha+\beta|^2} .$$

The compact picture for the principal series can now be made explicit starting from (3.6) and (3.7). The resulting action $\pi_{\varepsilon,\lambda}$ of $x \in G$ on $F \in \mathcal{H}^{\varepsilon}$ is given by

$$(x \cdot F)(e^{i\theta}) = |\bar{\alpha}e^{i\theta} - \beta e^{-i\theta}|^{-(\lambda+1)} F\left(\frac{\bar{\alpha}e^{i\theta} - \beta e^{-i\theta}}{|\bar{\alpha}e^{i\theta} - \beta e^{-i\theta}|}\right).$$

A convenient transformation is to identify the $\mathcal{H}^{\varepsilon}$ with $L^2(K)$ in the following manner. $F \in \mathcal{H}^{\varepsilon}$ corresponds to $f \in L^2(K)$, the defining equation being $f(e^{i\theta}) = F(e^{i\theta/2})$ if $\varepsilon = 0$ and $f(e^{i\theta}) = e^{i\theta/2}F(e^{i\theta/2})$ if $\varepsilon = 1$. Then the above action can be rewritten as

$$(x \cdot f)(e^{i\theta}) = \left(\bar{\alpha} - \beta e^{-i\theta}\right)^{\tau - \frac{\varepsilon}{2}} \left(\alpha - \bar{\beta}e^{i\theta}\right)^{\tau + \frac{\varepsilon}{2}} f\left(\frac{\bar{\alpha} - \beta e^{-i\theta}}{\alpha e^{-i\theta} - \bar{\beta}}\right). \tag{3.11}$$

where $\tau = -\frac{\lambda+1}{2}$, $f \in L^2(K)$. Now we are in a position to calculate the matrix coefficients of the unitary principal series. Recall from §3.4 the basis $\{e_m\}_{m\in\mathbb{Z}}$ for $L^2(K)$. Then the action of $x \in G$ on e_m is given by

$$(x \cdot e_m)(e^{i\theta}) = \left(\bar{\alpha} - \beta e^{-i\theta}\right)^{\tau + m - \frac{\varepsilon}{2}} \left(\alpha - \bar{\beta}e^{i\theta}\right)^{\tau - m + \frac{\varepsilon}{2}} e^{im\theta}.$$

Therefore the matrix coefficients are given by

$$\begin{aligned} \varphi_{m,n}(x) &= \langle x \cdot e_m, e_n \rangle \\ &= \int_0^{2\pi} \left(\bar{\alpha} - \beta e^{-i\theta}\right)^{\tau + m - \frac{\varepsilon}{2}} \left(\alpha - \bar{\beta}e^{i\theta}\right)^{\tau - m + \frac{\varepsilon}{2}} e^{i(m-n)\theta} d\theta. \end{aligned} \tag{3.12}$$

Apply the binomial theorem to the terms of the integrand to get:

$$\begin{aligned} \left(\bar{\alpha} - \beta e^{-i\theta}\right)^{\tau + m - \frac{\varepsilon}{2}} &= \bar{\alpha}^{\tau + m - \frac{\varepsilon}{2}} \sum_{k=0}^{\infty} \frac{\Gamma(\tau + m - \frac{\varepsilon}{2} + 1)}{k!\Gamma(\tau + m - k - \frac{\varepsilon}{2} + 1)} \left(\frac{-\beta e^{-i\theta}}{\bar{\alpha}}\right)^k, \\ \left(\alpha - \bar{\beta}e^{i\theta}\right)^{\tau - m + \frac{\varepsilon}{2}} &= \alpha^{\tau - m + \frac{\varepsilon}{2}} \sum_{l=0}^{\infty} \frac{\Gamma(\tau - m + \frac{\varepsilon}{2} + 1)}{l!\Gamma(\tau - m - l + \frac{\varepsilon}{2} + 1)} \left(\frac{-\bar{\beta}e^{i\theta}}{\alpha}\right)^l. \end{aligned}$$

Substituting these expressions into (3.11) and noting that the integral of any term in the expansion will be non-zero exactly when $k - l = m - n$, we find

$$\begin{aligned} \varphi_{m,n}(x) &= (-1)^{m-n}\ \bar{\alpha}^{\tau + m - \frac{\varepsilon}{2}}\ \alpha^{\tau - n + \frac{\varepsilon}{2}}\ \bar{\beta}^{n-m} \\ \times &\quad \Gamma(\tau + m - \frac{\varepsilon}{2} + 1)\ \Gamma(\tau - m + \frac{\varepsilon}{2} + 1) \\ \times &\quad \sum_{s=N}^{\infty} \frac{|\beta/\alpha|^{2s}}{s!\,(n-m+s)!\,\Gamma(\tau + m - s - \frac{\varepsilon}{2} + 1)\,\Gamma(\tau - n - s + \frac{\varepsilon}{2} + 1)} \end{aligned} \tag{3.13}$$

[5]Note that this decomposition is not obtained by conjugating the one described for $SL(2,\mathbb{R})$ by T, but arises from a different choice of "A". The choices involved in obtaining an Iwasawa decomposition are described in §5.1.

where $N = \max\{0, m-n\}$. Using the definition of the hypergeometric function and the properties of the gamma function, given in §A.2, we find

$$\begin{aligned}\varphi_{m,n}(x) &= (-1)^{m-n}\frac{\bar\alpha^{\tau+m-\frac{\varepsilon}{2}}\ \alpha^{\tau-n+\frac{\varepsilon}{2}}\ \bar\beta^{n-m}\ \Gamma(\tau-m+\frac{\varepsilon}{2}+1)}{(n-m)!\ \Gamma(\tau-n+\frac{\varepsilon}{2}+1)}\\ &\times\ F\left(-\tau+n-\frac{\varepsilon}{2}, -\tau-m+\frac{\varepsilon}{2}; n-m+1; \left|\frac{\beta}{\alpha}\right|^2\right), \quad (3.14)\end{aligned}$$

for $m \le n$, and

$$\begin{aligned}\varphi_{m,n}(x) &= (-1)^{m-n}\frac{\bar\alpha^{\tau+n-\frac{\varepsilon}{2}}\ \alpha^{\tau-m+\frac{\varepsilon}{2}}\ \beta^{m-n}\ \Gamma(\tau+m-\frac{\varepsilon}{2}+1)}{(m-n)!\ \Gamma(\tau+n-\frac{\varepsilon}{2}+1)}\\ &\times\ F\left(-\tau+m-\frac{\varepsilon}{2}, -\tau-n+\frac{\varepsilon}{2}; m-n+1; \left|\frac{\beta}{\alpha}\right|^2\right). \quad (3.15)\end{aligned}$$

for $m \ge n$. Let $x = e_\phi x_t e_\psi$ be the polar decomposition (3.2). Then we find

$$\varphi_{m,n}(x) = e^{-i[(2m-\varepsilon)\psi+(2n-\varepsilon)\phi]}\varphi_{m,n}(x_t). \quad (3.16)$$

So for understanding the asymptotic behaviour of the matrix coefficients, we need to study the function $\Phi_{m,n}(t) = \varphi_{m,n}(x_t)$. Applying (3.14), we see that

$$\begin{aligned}\Phi_{m,n}(t) &= (-1)^{m-n}\frac{(\cosh t)^{2\tau}\ (\tanh t)^{m-n}\Gamma(\tau+m-\frac{\varepsilon}{2}+1)}{(m-n)!\ \Gamma(\tau+n-\frac{\varepsilon}{2}+1)}\\ &\times\ F\left(-\tau+m-\frac{\varepsilon}{2}, -\tau-n+\frac{\varepsilon}{2}; m-n+1; \tanh^2 t\right). \quad (3.17)\end{aligned}$$

for $m \ge n$. Comparing with (3.13) we see that the formula for the $m < n$ case is obtained by replacing (m,n) with $(-m,-n)$ on the right side of (3.15).

Theorem 3.7.1 *The matrix coefficients $\varphi_{m,n}$ of the unitary principal series are not square integrable. But they are in $L^p(G)$ for every $p > 2$.*

Proof. According to (3.15) the decay of the matrix coefficients is measured by $(\cosh t)^{2\tau}$. Consider the unitary principal series, so that $2\tau = -(i\rho+1)$, $\rho \in \mathbb{R}$. By Lemma 3.1.1 and the fact that $\sinh 2t$ is asymptotically e^{2t}, while $\cosh t$ is e^t, we see that $\varphi_{m,n} \in L^p(G)$ exactly when

$$\int_0^\infty \left|e^{-t(i\rho+1)}\right|^p e^{2t}\,dt = \int_0^\infty e^{-t(p-2)}\,dt < \infty.$$

□

The calculations in this section closely follow those in the book of Vilenkin and Klimyk [75], differing only in superficial changes of parameters. [75] and its companion volumes examine in detail the relationship between group representations and special functions.

3.8 Complementary Series

Recall that for $-1 < \lambda < 0$, $\pi_{0,\lambda}$ has underlying $(\mathfrak{g}, K)$ module $P(0 : \lambda^2)$, a member of the complementary series. This suffices to calculate the matrix coefficients of the complementary series. Let us use the realization (3.11) of the induced representations. Then we get the following actions of the Lie algebra elements H, X, and Y (we mean by these the conjugates of the elements of the same names in $\mathfrak{sl}(2,\mathbb{R})$):

$$\begin{aligned}
H &: e_m \mapsto 2me_m, \\
X &: e_m \mapsto ime_m - \frac{1}{2}((\tau+m)e_{m-1} + (\tau-m)e_{m+1}), \\
Y &: e_m \mapsto -ime_m - \frac{1}{2}((\tau+m)e_{m-1} + (\tau-m)e_{m+1}),
\end{aligned}$$

where $\tau = -(\lambda+1)/2$. Now we can repeat our calculations of §3.3 for the unitarizability conditions. We find the following relationship is necessary and sufficient for an inner product on the induced space to give a unitary representation:

$$(\tau+n+1)||e_n||^2 = -(\tau-n)||e_{n+1}||^2.$$

After a scaling, we get the following inner product:

$$\langle e_m, e_n\rangle = \begin{cases} 0 & \text{if } n \neq m \\ \Gamma(\tau+|m|)/\Gamma(-\tau+|m|) & \text{if } n = m \end{cases}. \tag{3.18}$$

(The scaling of $||\cdot||^2$ is by a factor of $\Gamma(-\tau)/\Gamma(\tau+1)$, which is seen to be a positive real number by the reflection formula A.3.) Completing the span of $\{e_m\}_{m=0}^{\infty}$ with respect to the inner product (3.17) we get a unitary representation of $SU(1,1)$ which we call C^λ. Its matrix coefficients

$$\theta_{m,n}(x) = \langle x\cdot\frac{e_m}{||e_m||}, \frac{e_n}{||e_n||}\rangle$$

can then be found:

$$\theta_{m,n}(x) = \left[\frac{\Gamma(-\tau+|m|)\Gamma(\tau+|n|)}{\Gamma(\tau+|m|)\Gamma(-\tau+|n|)}\right]^{1/2}\varphi_{m,n}(x), \tag{3.19}$$

where $\varphi_{m,n}$ is given by the formulas (3.13) and (3.14). Thus the asymptotic behaviour is again given by $\Phi_{m,n}(t)$.

Theorem 3.8.1 *For any complementary series representation C^λ, $-1 < \lambda < 0$, $\exists\epsilon > 0$ such that the matrix coefficients $\theta_{m,n}$ are not in $L^{2+\epsilon}(G)$.*

Proof. $-1 < \lambda < 0$ implies $-1/2 < \tau < 0$. Let $\epsilon > 0$. Then the condition for $\theta_{m,n}$ to be in $L^{2+\epsilon}(G)$ is that

$$\int_0^\infty (e^{2t\tau})^p e^{2t}dt < \infty.$$

This is equivalent to

$$\tau < \frac{-1}{2+\epsilon}$$

which is clearly violated for small enough ϵ. $\square$

Another realization of the complementary series arises from the *non-compact picture* of induction from B. It is easy to see that N^tB is open in B with a complement of lower dimension. This can be used to realize the induced representation in functions on $N^t \cong \mathbb{R}$. The non-compact picture is described in general in chapter five, and in particular for $SL(2,\mathbb{C})$ in §4.1. So we do not give the calculations here, but go straight to the final result:

Consider the following norm on measurable functions on $\mathbb{R}$:

$$\langle f_1|f_2\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^{-\lambda-1} f_1(x) f_2(y)\, dx\, dy$$

Take the Hilbert space consisting of those f for which $\langle f|f\rangle < \infty$. Then $SL(2,\mathbb{R})$ acts unitarily on it as follows:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : f(x) \mapsto |bx+d|^{\lambda-1} f\left(\frac{ax+c}{bx+d}\right)$$

This is the complementary series representation C^λ. Observe that at $\lambda = -1$ the inner product becomes positive semi-definite, and dividing by the kernel leaves the trivial one dimensional representation. So in a sense the trivial representation is a limit of the complementary series.

3.9 Topology of Unitary Dual

We sum up the situation for $SL(2,\mathbb{R})$ by giving a description of the topologies of the unitary dual and the reduced dual. We had concluded §1.5 with some remarks about the possible meaning of the word "character" for non-compact groups. We now give a more concrete picture for $SL(2,\mathbb{R})$. The space $C_c^\infty(G)$ of smooth compactly supported functions is dense in $C^*(G)$. We show below that it allows us to define the character of a representation.

Theorem 3.9.1 *Let $(\pi,\mathcal{H})$ be an irreducible unitary representation of $SL(2,\mathbb{R})$ and $f \in C_c^\infty(G)$. Then $\pi(f)$ is a trace class operator.*

Proof. Let $\mathcal{H} = \sum_{n\in\mathbb{Z}} m_n V_n$ be the decomposition of $\mathcal{H}$ into K-types, with $V_n = \mathbb{C}$ being acted on by H' with eigenvalue n, and $m_n \in \{0,1\}$. Let $\{v_n\}$ be an orthonormal basis for $\mathcal{H}$. It suffices to show that $\sum_{m,n} |\langle \pi(f)v_m, v_n\rangle| < \infty$. We may assume that $v_n \in V_n$. Let E_n be the orthogonal projection onto V_n. Consider the element $z = (H')^2$ in the enveloping algebra of $\mathfrak{k}$. Then we have:

$$\begin{aligned}
\sum_{n,m} |\langle \pi(f)v_m, v_n\rangle| &= \sum_{n,m} |\langle E_n\pi(f)v_m, v_n\rangle| \\
&= \sum_{n,m} \frac{1}{n^2} |\langle \pi(z)E_n\pi(f)v_m, v_n\rangle| \\
&= \sum_{n,m} \frac{1}{n^2} |\langle E_n\pi(z)\pi(f)v_m, v_n\rangle| \\
&= \sum_{n,m} \frac{1}{n^2} |\langle E_n\pi(zf)v_m, v_n\rangle| \\
&= \sum_{n,m} \frac{1}{n^2} |\langle \pi(zf)v_m, v_n\rangle| \\
&\leq \sum_{n} \frac{1}{n^2} ||\pi(zf)|| \;<\; \infty.
\end{aligned}$$

Here zf is obtained by infinitesimal left translation, i.e. for $X \in \mathfrak{g}_o$ we define

$$(Xf)(x) = \frac{d}{dt}\Big|_{t=0} f(e^{-tX}x),$$

and extend this to a $\mathcal{U}(\mathfrak{g})$ action in the usual manner. □

The existence of characters has an immediate consequence for the topology of $\widehat{G}$.

Theorem 3.9.2 *$G = SL(2,\mathbb{R})$ is CCR (hence $\widehat{G}$ is T_1).*

Proof. Suppose $(\pi, \mathcal{H})$ is an irreducible unitary representation of G. Let $f \in C^*(G)$ and $\{f_\alpha\}$ be a net converging to f in the Fell topology, such that $f_\alpha \in C_c^\infty(G)$. We have

$$||\pi(f) - \pi(f_\alpha)|| = ||\pi(f - f_\alpha)|| \leq ||f - f_\alpha||,$$

where the last norm is the one for the Fell topology. This shows that $\pi(f_\alpha)$ converges to $\pi(f)$ in norm, and hence $\pi(f)$ is compact. □

Now we set up some notation. By $\mathcal{L}$ we mean the members of the unitary principal series, with $\pi_{\varepsilon,i\nu}$ and $\pi_{\varepsilon,-i\nu}$ identified with each other. The symbol $\mathcal{P}$ denotes the irreducible members of $\mathcal{L}$, and $\mathcal{R}$ is the set of members of $\widehat{G}$ which arise as direct summands of $\mathcal{L} \setminus \mathcal{P}$. The discrete series is labelled $\mathcal{D}$. More explicitly, we have:

$$\begin{aligned}
\mathcal{L} &= \{\pi_{\varepsilon,i\nu} :\ \varepsilon \in \{0,1\},\ \nu \in \mathbb{R}_{\geq 0}\} \\
\mathcal{P} &= \{\pi_{\varepsilon,i\nu} \in \mathcal{L} :\ (\varepsilon,\nu) \neq (1,0)\} \\
\mathcal{R} &= \{D(0:\pm)\} \\
\mathcal{D} &= \{D(-n:+), D(n:-) :\ n > 0\}
\end{aligned}$$

We can list the members of the reduced dual $\widehat{G}^r$:

Theorem 3.9.3 *$\widehat{G}^r = \mathcal{P} \cup \mathcal{R} \cup \mathcal{D}$ is a disjoint union.*

Proof. Theorems 1.5.6, 3.5.3, 3.6.1 and 3.7.1 imply that $\mathcal{P} \cup \mathcal{R} \cup \mathcal{D} \subset \widehat{G}^r$. Further, for any linear semisimple group G, it has been shown that the K-finite matrix coefficients of a member of $\widehat{G}^r$ are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$ [17, Thm 2]. This combines with Theorem 3.8.1 to show the complementary series are not in the reduced dual. $\square$

The following result about the reduced dual is taken from Lipsman [97], where the general situation of a semisimple group (and also the special case of a rank one group) is considered. The proof uses a description by Fell of convergence in the Fell topology through decomposition of characters, and some explicit information on characters in the reductive case that was given by Harish-Chandra. As we have not made any character calculations, we merely reproduce the result without proof.

We give $\mathcal{L}$ the topology induced from the set $\mathbb{Z}/2 \times \mathbb{R}$, and let $\mathcal{P}$ have the induced topology: we call this the natural topology of $\mathcal{P}$. For any $\mathcal{S} \subset \mathcal{P}$, we denote by $\tilde{\mathcal{S}}$ its closure in $\mathcal{P}$ in the natural topology, and by $\bar{\mathcal{S}}$ its closure in $\widehat{G}^r$ in the Fell topology. ($\widehat{G}^r$ is closed in $\widehat{G}$.)

Theorem 3.9.4

1. $\mathcal{P} \cup \mathcal{R}$ and $\mathcal{D}$ are open in $\widehat{G}^r$.

2. The Fell topology of $\mathcal{D}$ is discrete.

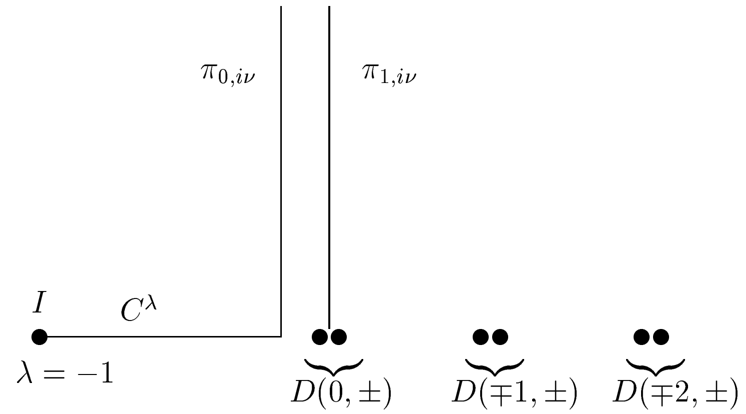
3. The Fell topology of $\mathcal{R}$ is discrete, while the Fell topology of $\mathcal{P}$ is the same as its natural topology.

4. Let $\mathcal{S} \subset \mathcal{P}$. If $\mathcal{S}$ contains a sequence $\pi_{1,i\nu}$ with $\nu \to 0$ then $\bar{\mathcal{S}} = \tilde{\mathcal{S}} \cup \mathcal{R}$. Else, $\bar{\mathcal{S}} = \tilde{\mathcal{S}}$. $\square$

Thus, the non-trivial part of $\widehat{G}$ may be represented by the following picture. (I is the trivial representation.) Its topology is that induced from the plane, with the following exceptions:

1. As $\nu \to 0$, $\pi_{1,i\nu} \to D(0, \pm)$.
2. As $\lambda \to -1$, $C^\lambda \to D(\mp 1, \pm), I$.

The last fact is mentioned in Lipsman [97] as "well-known", but appears to have been absent from the literature till Miličić [104].[6] It reflects the fact that $P(0:1)$ contains $D(\mp 1, \pm)$ as subrepresentations, and the quotient $P(0:1)/D(1,-) \oplus D(-1,+)$ is the trivial representation. The $D(\mp n, \pm)$ are paired to represent their common multiplicity in $L^2(G)$.



[6]Later, Kraljević and Miličić [91] described the C^*-algebra of $SL(2,\mathbb{R})$.