Chapter 4

Representations of $SL(2,\mathbb{C})$

In this chapter we describe the representation theory of $SL(2,\mathbb{C})$. The algebraic approach is more complicated here than for $SL(2,\mathbb{R})$. This is partly due to the doubling of dimension, and otherwise to the fact that the maximal compact subgroup $K = SU(2)$ is not abelian. Somewhat paradoxically, the eventual classification of irreducible unitary representations turns out to be simpler.

Let us recall how we tackled $SL(2,\mathbb{R})$. First, we used the algebraic approach to classify the irreducible $(\mathfrak{g}, K)$-modules. This involved writing down what the root spaces of $\mathfrak{g}_{\mathbb{C}}$ would do to the K-types. To find the unitary subset, we wrote down the conditions for the elements of $\mathfrak{g}$ to act as skew-Hermitian operators. Finally, we used the analytic method of induction to construct the principal series, and to see that all the irreducible representations are subquotients of these.

The same approach could be used for $SL(2,\mathbb{C})$. This is done explicitly in Taylor [128]. However, the algebraic approach involves a lot of manipulation and so we confine ourselves to the analytic approach. Essentially we present extracts from the original paper of Gelfand and Naimark [38].

As for $SL(2,\mathbb{R})$, we have an Iwasawa decomposition of $G = SL(2,\mathbb{C})$, and it is given by

$$\begin{aligned}
K &= SU(2), \\
A &= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \ t \in \mathbb{R}, \\
N &= \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \ z \in \mathbb{C}.
\end{aligned}$$

4.1 Principal Series

This starts off as a repeat of §3.4, with some minor differences. The new material starts with the discussion of the "noncompact picture".

Let B be the *Borel subgroup* of upper-triangular matrices. Corresponding to the Iwasawa decomposition $G = KAN$, there is the *Levy-Langlands decomposition* $B = MAN$, where $M =$ diagonal subgroup of $K \cong S^1$. Let χ be a quasi-character of B of the form $\chi = \xi \otimes \eta \otimes 1$ where ξ is a character of M, $\eta : A \to \mathbb{C}^\times$ is a quasi-character of A, and 1 denotes the trivial action of N on $\mathbb{C}$. Let $n \in \mathbb{Z}$ represent ξ, with

$$\xi \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{in\theta}.$$

The quasi-character η corresponds to a $\lambda \in \mathbb{C}$:

$$\eta \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = e^{\lambda t}.$$

Now we can describe $\pi_\chi = \pi_{n,\lambda}$, the representation of G induced from χ. Calculating as in §3.4 we find that

$$\triangle_G \equiv 1, \quad \triangle_B(man) = e^{4t}, \quad \text{where } a = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The space $C_c(G/B, \chi \otimes \triangle_{G/B}^{-1/2})$ is then given by the following invariance condition:

$$f(xb) = e^{-(\lambda+2)t(\log a)} \xi(m)^{-1} f(x) \ \forall b = man \in B, x \in G.$$

When η (and hence χ) is unitary the representation space $\mathcal{H}_\chi$ of π_χ is obtained by completing $C_c(G/B, \chi \otimes \triangle_{G/B}^{-1/2})$ in the usual way. The G action is

$$(x \cdot f)(g) = f(x^{-1} g).$$

Compact picture

Note that $G/B \cong K/M$. We can now represent $\mathcal{H}_\chi$ as functions on K with an invariance property with respect to M. Then, just as for $SL(2,\mathbb{R})$,

$$\mathcal{H}_\chi \cong \mathcal{H}^n = \left\{ f \in L^2(K) | \ f(km) = \xi(m)^{-1} f(k), k \in K, \ m \in M \right\}.$$

The identification of $\mathcal{H}_\chi$ with $\mathcal{H}^n$ is given by restriction to K. Calculating as for $SL(2,\mathbb{R})$, we get the action

$$(x \cdot f)(u) = e^{-(\lambda+2)t(x^{-1}u)} f(x^{-1}[u]).$$

As previously, the compact picture allows us to define the indeuced representations even when χ is non-unitary. In Chapter 3 we used the compact picture to connect the analytic constructions with the algebraic classification, and thus to determine the reducibility of the unitary principal series. Now we seek to show irreducibility directly, and for this it is more convenient to use a different description, which we give next.

Non-compact picture

Instead of restricting the induced picture to K, we now restrict it to $\bar{N} = N^t$. The motivation is that $\bar{N}B$ is an open dense submanifold of G — it consists of the matrices whose upper left entry is non-zero. So we can restrict first to $\bar{N}B$ and then to $\bar{N}$ without loss of information.

The representation space gets identified with $L^2(\bar{N})$, since $\bar{N}\cap K = \bar{N}\cap B = I$. The identification is given in one direction by restriction to $\bar{N}$. In the other direction $F \in L^2(\bar{N})$ corresponds to $f \in \mathcal{H}_\chi$ defined by

$$f(\bar{n}b) = \chi(b)^{-1}F(\bar{n}), \qquad (\bar{n} \in \bar{N}, b \in B)$$

on $\bar{N}B$. To get the G action on $F \in L^2(\bar{N})$, we first let G act on f, then restrict the result to $\bar{N}B$. Let us write $[\bar{n}b] = \bar{n}$. This gives

$$\begin{aligned}(x\cdot F)(\bar{n}) &= (x\cdot f)(\bar{n}) = \rho_x(\bar{n})^{1/2} f(x^{-1}\bar{n}) \\ &= \rho_x(\bar{n})^{1/2}\chi(b(x^{-1}\bar{n}))^{-1}F([x^{-1}\bar{n}]).\end{aligned}$$

This can be made very explicit. For an element of $\bar{N}B$, the decomposition[1] is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

Let us identify $\bar{N}$ with $\mathbb{C}$ in the obvious manner. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} d-bz & -b \\ az-c & a \end{pmatrix}$$

$$\Longrightarrow \quad [x^{-1}\bar{n}] = \frac{az-c}{d-bz}, \quad \text{where } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \bar{n} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

We also calculate the MAN decomposition:

$$\begin{aligned} b(x^{-1}\bar{n}) &= \begin{pmatrix} d-bz & -b \\ 0 & (d-bz)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{d-bz}{|d-bz|} & 0 \\ 0 & (\frac{d-bz}{|d-bz|})^{-1} \end{pmatrix}\begin{pmatrix} |d-bz| & 0 \\ 0 & |d-bz|^{-1} \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This gives

$$\chi(b(x^{-1}\bar{n}))^{-1} = \left(\frac{d-bz}{|d-bz|}\right)^{-n}|d-bz|^{-\lambda}, \quad \rho_x(\bar{n})^{1/2} = |d-bz|^{-2}.$$

The G action on $L^2(\bar{N}) = L^2(\mathbb{C})$ can now be written out:

$$\pi_{n,\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix}F(z) = |d-bz|^{n-2-\lambda}(d-bz)^{-n}F\left(\frac{az-c}{d-bz}\right). \tag{4.1}$$

Theorem 4.1.1 *The representations $\pi_{n,\lambda}$, $\lambda \in i\mathbb{R}$, are all irreducible and unitary.*

Proof. Since $\lambda \in i\mathbb{R}$, χ is a unitary representation of B, and $\pi_{n,\lambda}$ is a unitary representation of $SL(2,\mathbb{C})$. Now suppose there is a subrepresentation. Then projection onto the invariant subspace is a bounded operator which commutes with all the $\pi_{n,\lambda}(x)$. We will show that any operator R which commutes with all $\pi_{n,\lambda}(b)$, $b \in B^t$, is a scalar, so that $\pi_{n,\lambda}$ is in fact irreducible on B^t. Let us use the noncompact picture (4.1).

[1]Note the unfortunate use of b for both a matrix and a matrix entry.

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} & \Longrightarrow (b_1 \cdot F)(z) = F(z+u), \\ b_2 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} & \Longrightarrow (b_2 \cdot F)(z) = |a|^{2+\lambda-n} a^n F(a^2 z). \end{aligned}$$

Consider the Fourier transform on $L^2(\mathbb{C})$:

$$\hat{F}(w) = \frac{1}{2\pi} \int_{\mathbb{C}=\mathbb{R}^2} F(z) e^{-i\mathrm{Re}(\bar{z}w)}\, d\mu(z), \quad z, w \in \mathbb{C}, \tag{4.2}$$

where $d\mu(z) = dz\, d\bar{z}$ is the Lebesgue measure on $\mathbb{C}$. The transform $F \mapsto \hat{F}$ is an isometry of $L^2(\mathbb{C})$ onto itself. We have the following easy formulae:

$$(b_1 \cdot F)\hat{}(w) = e^{i\mathrm{Re}(\bar{u}w)} \hat{F}(w), \tag{4.3}$$

$$(b_2 \cdot F)\hat{}(w) = |a|^{2+\lambda-n} a^n \hat{F}(\bar{a}^{-2} w). \tag{4.4}$$

Define the operator $\hat{R}$ by $\hat{R}\hat{F} = (RF)\hat{}$. For a bounded Borel function f on $\mathbb{C}$, the multiplication operator $M_f : \phi \mapsto f\phi$ is a bounded operator on $L^2(\mathbb{C})$. Denote $\alpha(w) = e^{i\mathrm{Re}(\bar{u}w)}$. Then formula (4.3) implies that $\hat{R}M_\alpha = M_\alpha \hat{R}$, and from this it follows that $\hat{R}$ commutes with every M_f, $f \in L^\infty(\mathbb{C})$. Consequently, $\hat{R} = M_\phi$ for some bounded Borel function ϕ [119, p325]. Formula (4.4) now shows that $\phi(w) = \phi(\bar{a}^{-2}w)$ for almost all w. Therefore ϕ is constant almost everywhere, and $\hat{R}$ is a scalar. Hence so is R. □

The representations $\pi_{n,\lambda}$, $\lambda \in i\mathbb{R}$, are called the *unitary principal series* of $SL(2,\mathbb{C})$. The full *principal series* consists of all the $\pi_{n,\lambda}$, $\lambda \in \mathbb{C}$.

Theorem 4.1.2 *The unitary principal series representations $\pi_{n,i\nu}$ and $\pi_{-n,-i\nu}$ are unitarily equivalent.*

Proof. We will exhibit an intertwining operator for the two representations. As in the previous theorem it is convenient to first apply the Fourier transform. Thus, on $L^2(\mathbb{C})$, we define a representation $\hat{\pi}_{n,i\nu}$ by $\hat{\pi}_{n,i\nu}(g)\hat{F} = (\pi_{n,i\nu}(g)F)\hat{}$. This representation is equivalent to $\pi_{n,i\nu}$ (the Fourier transform gives the equivalence).

We wish to find a unitary operator V such that

$$V \circ \hat{\pi}_{-n,-i\nu}(g) = \hat{\pi}_{n,i\nu}(g) \circ V, \qquad \forall g \in G. \tag{4.5}$$

For $g = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$, equation (4.3) implies

$$(\hat{\pi}_{n,i\nu}(g)\hat{F})(w) = e^{i\mathrm{Re}\,(z\bar{w})} \hat{F}(w),$$

and putting this in (4.5) we get

$$V e^{i\mathrm{Re}\,(z\bar{w})} \hat{F}(w) = e^{i\mathrm{Re}\,(z\bar{w})} (V\hat{F})(w).$$

So V commutes with multiplication by $e^{i\mathrm{Re}\,(z\bar{w})}$ and hence is given by multiplication by a bounded function ϕ: $VF(w) = \phi(w)F(w)$. For $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, equation (4.4) implies

$$(\hat{\pi}_{n,i\nu}(g)\hat{F})(w) = |a|^{2+i\nu-n} a^n \hat{F}(\bar{a}^{-2} w),$$

and then equation (4.5) gives

$$\phi(\bar{a}^{-2}w) = a^{-2n}|a|^{2(n-i\nu)}\phi(w).$$

Replacing w by 1 and $\bar{a}^{-2}$ by ζ, we get

$$\phi(\zeta) = \bar{\zeta}^n|\zeta|^{i\nu-n}.$$

(The constant term $\phi(1)$ on the right side can be taken to be 1.) We have found a candidate intertwining operator. To see it works let's revert to the original $\pi_{n,i\nu}$. If we denote the intertwining operator there by T then we have $TF = (V\hat{F})\check{}$ where $\tilde{f}$ denotes the inverse Fourier transform of f. The explicit formula for T is therefore:

$$TF(z) = \frac{1}{2\pi}\int_{\mathbb{C}}\int_{\mathbb{C}} \bar{w}^n|w|^{i\nu-n}e^{i\mathrm{Re}\,(z-\zeta)\bar{w}}F(\zeta)\,d\mu(w)\,d\mu(\zeta).$$

It is straightforward to check that this operator intertwines the two actions. Moreover V is an isometry since $|\phi| \equiv 1$, and hence T is also an isometry. Finally, since the representations concerned are irreducible, T is an equivalence.

□

4.2 Complementary Series

Recall that for $SL(2,\mathbb{R})$ the complementary series representations were infinitesimally equivalent to the principal series representations $\pi_{0,\lambda}$, $\lambda \in (0,1)$, but with a different norm. Similarly, we can obtain a complementary series for $SL(2,\mathbb{C})$. Consider the noncompact picture for $\pi_{n,\lambda}$, and let us attempt to find a metric of the form

$$\langle F_1, F_2\rangle = \int_{\mathbb{C}}\int_{\mathbb{C}} K(z_1,z_2)F_1(z_1)\overline{F(z_2)}\,d\mu(z_1)\,d\mu(z_2), \tag{4.6}$$

where $K(z_1,z_2)$ is a measurable function, and such that the action (4.1) preserves the metric. In order to get a unitary action, we need:

$$\begin{aligned}\int_{\mathbb{C}}\int_{\mathbb{C}} & K(z_1,z_2)F_1(z_1)\overline{F_2(z_2)}\,d\mu(z_1)\,d\mu(z_2) = \\ & \int_{\mathbb{C}}\int_{\mathbb{C}} K(z_1,z_2)|d-bz_1|^{n-2-\lambda}(d-bz_1)^{-n}|d-bz_2|^{n-2-\lambda}\overline{(d-bz_2)}^{-n} \\ & \qquad \times F_1\left(\frac{az_1-c}{d-bz_1}\right)\overline{F_2\left(\frac{az_2-c}{d-bz_2}\right)}\,d\mu(z_1)\,d\mu(z_2).\end{aligned}$$

Carrying out the change of variable $z_i \to (az_i-c)/(d-bz_i)$, $i=1,2$, on the left side of the above equation, we find that K should satisfy the following identity:

$$\begin{aligned}K&\left(\frac{az_1-c}{d-bz_1},\frac{az_2-c}{d-bz_2}\right) = \\ &K(z_1,z_2)|d-bz_1|^{n+2-\lambda}|d-bz_2|^{n+2-\lambda}(d-bz_1)^{-n}\overline{(d-bz_2)}^{-n}.\end{aligned}$$

Now consider $g = \left(\begin{smallmatrix} 1 & 0 \\ \zeta & 1 \end{smallmatrix}\right)$. [2] Then we get

$$K(z_1 - \zeta, z_2 - \zeta) = K(z_1, z_2),$$

and hence K is of the form $K(z_1, z_2) = K_1(z_1 - z_2)$ for some function K_1 of one variable. For K_1 the equation becomes

$$K_1\left(\frac{z_1 - z_2}{(d - bz_1)(d - bz_2)}\right) = \\ K_1(z_1 - z_2)|d - bz_1|^{n+2-\lambda}|d - bz_2|^{n+2-\lambda}(d - bz_1)^{-n}\overline{(d - bz_2)}^{-n}.$$

Substituting $z_1 = 0$ and choosing b such that $d - bz_2 = 1$, we get

$$K_1\left(-\frac{z_2}{d}\right) = K_1(-z_2)|d|^{n+2-\lambda}d^{-n}.$$

Similarly, we also get

$$K_1\left(\frac{z_1}{d}\right) = K_1(z_1)|d|^{n+2-\lambda}\bar{d}^{-n}.$$

On comparing the last two equations, we see that we require $n = 0$. So now we have

$$K_1\left(\frac{z}{d}\right) = K_1(z)|d|^{2-\lambda},$$

and hence

$$K_1(z) = C\frac{1}{|z|^{2-\lambda}}.$$

Evidently, in order for (4.5) to satisfy $\langle F, F\rangle \geq 0$, we also require $\lambda \in \mathbb{R}$ and $C > 0$. Thus our candidate metric is

$$\langle F_1, F_2\rangle = \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{F_1(z)\overline{F_2(\zeta)}}{|z - \zeta|^{2-\lambda}}\, d\mu(z)\, d\mu(\zeta) \qquad (\lambda \in \mathbb{R}). \tag{4.7}$$

What remains to be determined is the precise function space on which this metric makes sense, and the values of λ for which it is positive-definite. Recall the Fourier transform from (4.2). Consider the space $\mathcal{H}' = C_c^\infty(G)$ of smooth functions on $\mathbb{C}$ with compact support. The Fourier transform maps $\mathcal{H}'$ into the continuous L^1 functions (in fact into the smooth rapidly decreasing functions [127, ChIII]). [3] For $F \in \mathcal{H}'$ we have the inversion formula

$$F(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \hat{F}(w)e^{i\mathrm{Re}\,(z\bar{w})}\, d\mu(w). \tag{4.8}$$

[2] Suppose we look for a simpler metric, of the form $\int_{\mathbb{C}} K(z)F_1(z)\overline{F_2(z)}\, d\mu(z)$. Then the requirement for the action of $\left(\begin{smallmatrix} 1 & 0 \\ \zeta & 1 \end{smallmatrix}\right)$ to be unitary is easily seen to imply that $K \equiv$ const, and we do not get any new unitary representations. Note also that the L^2 metric can be written as $\int\int \delta(z - \zeta)F_1(z)\overline{F_2(\zeta)}\, d\mu(z)\, d\mu(\zeta)$, where δ is the Dirac distribution.

[3] A smooth function f is said to be rapidly decreasing if, as $z \to \infty$, it goes to 0 faster than any $|z|^{-n}$, $n \in \mathbb{N}$.

Now we start our investigation of (4.7) for $F_1 = F_2 = F \in \mathcal{H}'$. First we consider the convergent integral

$$\begin{aligned}
\int_{\mathbb{C}} |z-\zeta|^{\lambda-2} F(z)\, d\mu(z) &= \int_{\mathbb{C}} |z|^{\lambda-2} F(z+\zeta)\, d\mu(z) \\
&= \frac{1}{2\pi} \int_{\mathbb{C}} |z|^{\lambda-2} \left[\int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,((z+\zeta)\bar{w})}\, d\mu(w) \right] d\mu(z) && (4.9) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r^{\lambda-1} \left[\int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,(\zeta\bar{w})} e^{i\mathrm{Re}\,(rse^{i(\theta-\phi)})}\, d\mu(w) \right] dr\, d\theta && (4.10) \\
&= \frac{1}{2\pi} \int_0^{\infty} r^{\lambda-1} \left[\int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,(\zeta\bar{w})} \Big(\int_0^{2\pi} e^{i\mathrm{Re}\,(rse^{i(\theta-\phi)})}\, d\theta \Big)\, d\mu(w) \right] dr \\
&= \int_0^{\infty} r^{\lambda-1} \left[\int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,(\zeta\bar{w})} J_0(rs)\, d\mu(w) \right] dr \\
&= \int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,(\zeta\bar{w})} \left[\int_0^{\infty} r^{\lambda-1} J_0(rs)\, dr \right] d\mu(w) \\
&= C_\lambda \int_{\mathbb{C}} \hat{F}(w) e^{i\mathrm{Re}\,(\zeta\bar{w})} |w|^{-\lambda}\, d\mu(w) \qquad (0 < \lambda < \tfrac{1}{2}). && (4.11)
\end{aligned}$$

The many switches of order of integration above are justified by Fubini's theorem,[4] whose use is validated by our assumptions about F. Equality (4.9) uses the inversion formula (4.8). To get (4.10) we make the change of variable $z = re^{i\theta}$, and also write $w = se^{i\phi}$. The function J_0 is the Bessel function of order 0 (§A.3), and (4.11) follows from formula (A.6). The constant in (4.11) is

$$C_\lambda = 2^{\lambda-1} \frac{\Gamma(\lambda/2)}{\Gamma(1-\lambda/2)}.$$

Multiplying both sides of the final result by $\overline{F(\zeta)}$ and integrating, we get

$$\begin{aligned}
\langle F, F \rangle &= C_\lambda \int_{\mathbb{C}} \int_{\mathbb{C}} \hat{F}(w) \overline{F(\zeta)} e^{i\mathrm{Re}\,(\zeta\bar{w})} |w|^{-\lambda}\, d\mu(w)\, d\mu(\zeta) \\
&= 2^\lambda \pi \frac{\Gamma(\lambda/2)}{\Gamma(1-\lambda/2)} \int_{\mathbb{C}} |\hat{F}(w)|^2 |w|^{-\lambda}\, d\mu(w), && (4.12)
\end{aligned}$$

for $0 < \lambda < 1/2$. However, each side of this equation is analytic in λ in the strip $0 < \mathrm{Re}\,\lambda < 2$ and so the equality must hold throughout this strip. In particular it holds for $0 < \lambda < 2$. Hence (4.7) gives a positive definite metric for $0 < \lambda < 2$. Let $\mathcal{H}_\lambda$ be the corresponding completion of $\mathcal{H}'$.

In the rest of this section we always assume $\lambda \in (0, 2)$. Define

$$\mathfrak{H}' = \{ f = \hat{F} \mid F \in \mathcal{H}' \}$$

with inner product

$$\langle f_1, f_2 \rangle = 2^\lambda \pi \frac{\Gamma(\lambda/2)}{\Gamma(1-\lambda/2)} \int_{\mathbb{C}} |w|^{-\lambda} f_1(w) \overline{f_2(w)}\, d\mu(w). \qquad (4.13)$$

[4] Let (X, μ) and (Y, ν) be σ-finite measure spaces, and $\lambda = \mu \times \nu$ the product measure on $X \times Y$. Then Fubini's theorem states that (1) If h is an integrable function on $X \times Y$, then almost every section of h is integrable, the functions $x \mapsto \int_Y h(x, y)\, d\nu(y)$ and $y \mapsto \int_X h(x, y)\, d\mu(x)$ are integrable, and $\int_{X\times Y} h\, d\lambda = \int_Y \left[\int_X h\, d\mu\right] d\nu = \int_X \left[\int_Y h\, d\nu\right] d\mu$; (2) If h is complex and $\int_X \int_Y |h|\, d\nu(y)\, d\mu < \infty$ then $h \in L^1(\lambda)$.

Then $F \mapsto \hat{F}$ is an isometry of $\mathcal{H}'$ onto $\mathfrak{H}'$, and extends to an isometry of $\mathcal{H}_\lambda$ onto the completion $\mathfrak{H}_\lambda$ of $\mathfrak{H}'$.

Lemma 4.2.1 $\mathfrak{H}_\lambda = \{f : \mathbb{C} \xrightarrow{\text{mble}} \mathbb{C} \mid \int_{\mathbb{C}} |w|^{-\lambda}|f(w)|^2\, d\mu(w) < \infty\}$.

Proof. Let us first write $\tilde{\mathfrak{H}}_\lambda = \{f : \mathbb{C} \xrightarrow{\text{mble}} \mathbb{C} \mid \int_{\mathbb{C}} |w|^{-\lambda}|f(w)|^2\, d\mu(w) < \infty\}$. Now consider a compact subset Q of $\mathbb{C}$ not containing 0 and f_Q a measurable function supported on Q. Such f_Q are dense in the L^2 space $\tilde{\mathfrak{H}}_\lambda$. We will show that they are approximated by members of $\mathfrak{H}'$.

Since $\mathcal{H}'$ is dense in $L^2(\mathbb{C})$, so is $\mathfrak{H}'$. Hence for any $\varepsilon > 0$ there exists $f_\varepsilon \in \mathfrak{H}'$ such that $\int |f_Q - f_\varepsilon|^2\, d\mu < \varepsilon$. Fix C such that $|f_Q(z) - f_\varepsilon(z)| < C$ everywhere. Now take an arbitrary $\delta > 0$, and let $B_1 = \{w \ : \ |w| \le \delta\}$, $B_2 = B_1^c$. Then

$$\begin{aligned}
&\int_{\mathbb{C}} |w|^{-\lambda}|f_Q(w) - f_\varepsilon(w)|^2\, d\mu(w)\\
&\qquad = \int_{B_1} |w|^{-\lambda}|f_Q(w) - f_\varepsilon(w)|^2\, d\mu(w) \ + \int_{B_2} |w|^{-\lambda}|f_Q(w) - f_\varepsilon(w)|^2\, d\mu(w)\\
&\qquad \le C^2 \int_0^{2\pi}\int_0^\delta r^{1-\lambda}\, dr\, d\theta \ + \ \delta^{-\lambda}\int_{B_2} |f_Q(w) - f_\varepsilon(w)|^2\, d\mu(w)\\
&\qquad \le \frac{2\pi C^2}{2-\lambda}\delta^{2-\lambda} \ + \ \delta^{-\lambda}\varepsilon.
\end{aligned}$$

Elementary calculus shows that the last expression has a minimum value of the form $M(\lambda)C^\lambda\sqrt{\varepsilon}^{2-\lambda}$. As $\varepsilon \to 0$ we can arrange to have C decrease monotonically, and so $f_\varepsilon \to f_Q$ in the $\tilde{\mathfrak{H}}_\lambda$ norm. Hence $\mathfrak{H}'$ is dense in $\tilde{\mathfrak{H}}_\lambda$, and $\mathfrak{H}_\lambda = \tilde{\mathfrak{H}}_\lambda$.

□

We would like to have a similar description of the completion $\mathcal{H}_\lambda$. Let us define

$$\mathcal{H}'_\lambda = \left\{F : \mathbb{C} \xrightarrow{\text{mble,bdd}} \mathbb{C} \mid \int\int |z-\zeta|^{\lambda-2}|F(z)F(\zeta)|\, d\mu(z)\, d\mu(\zeta) < \infty\right\},$$

and put the norm $\int\int |z-\zeta|^{\lambda-2}F(z)\overline{F(\zeta)}\, d\mu(z)\, d\mu(\zeta)$ on it.

Lemma 4.2.2 *Let $F \in \mathcal{H}'_\lambda$. Then the integral*

$$\hat{F}(w) = \frac{1}{2\pi}\int_{\mathbb{C}} F(z)e^{-i\mathrm{Re}\,(\bar{z}w)}\, d\mu(z)$$

converges in the $\mathfrak{H}_\lambda$ norm and

$$||F||^2 = ||\hat{F}||^2,$$

where the left norm is in $\mathcal{H}'_\lambda$ and the right norm in $\mathfrak{H}_\lambda$.

Proof. For the first claim we need to show that for compact Q, the Fourier transform of the truncation of F to Q, denoted $\hat{F}_Q$, is in $\mathfrak{H}_\lambda$ and as $Q \uparrow \mathbb{C}$, the functions $\hat{F}_Q$ converge in $\mathfrak{H}_\lambda$ to a limit that we denote by $\hat{F}$.

Consider a bounded measurable function F with support in $\{z : |z| \leq C\}$. Then we can choose from $\mathcal{H}'$ a uniformly bounded sequence F_m such that $\operatorname{supp}(F_m) \subset \{z : |z| \leq C\}$ and $\int |F - F_m|^2 \, d\mu \to 0$ as $m \to \infty$. This implies that $\int |F - F_m| \, d\mu \to 0$. Let C' be a common bound for $|F|$ and all the $|F_m|$, and for $\varepsilon > 0$ define

$$\begin{aligned} Q_1 &= \left\{(z,\zeta) \;\middle|\; |z-\zeta| \leq \varepsilon\right\}, \\ Q_2 &= \left\{(z,\zeta) \;\middle|\; |z-\zeta| > \varepsilon\right\}. \end{aligned}$$

Now we have

$$\begin{aligned} &\left| \int_{Q_1} |z-\zeta|^{\lambda-2} [F(z) - F_m(z)] \overline{F(\zeta)} \, d\mu(z) \, d\mu(\zeta) \right| \\ &\qquad \leq 2C' \int_{Q_1} |z-\zeta|^{\lambda-2} |F(\zeta)| \, d\mu(z) \, d\mu(\zeta) \\ &\qquad \leq 2C' \int_{\mathbb{C}} \int_{|z| \leq \varepsilon} |z|^{\lambda-2} |F(\zeta) \, d\mu(z) \, d\mu(\zeta) \\ &\qquad \leq 2C'\pi C^2 \int_0^{2\pi} \int_0^{\varepsilon} r^{\lambda-1} \, dr \, d\theta = \frac{4\pi C'^2 C^2}{\lambda} \varepsilon^{\lambda}. \end{aligned}$$

And also

$$\begin{aligned} &\left| \int_{Q_2} |z-\zeta|^{\lambda-2} [F(z) - F_m(z)] \overline{F(\zeta)} \, d\mu(z) \, d\mu(\zeta) \right| \\ &\qquad \leq \varepsilon^{\lambda-2} \pi C^2 C' \int |F(z) - F_m(z)| \, d\mu(z). \end{aligned}$$

$$\begin{aligned} \text{Thus :} \quad & \int_{\mathbb{C}} |z-\zeta|^{\lambda-2} [F(z) - F_m(z)] \overline{F(\zeta)} \, d\mu(z) \, d\mu(\zeta) \to 0. \\ \text{Similarly :} \quad & \int_{\mathbb{C}} |z-\zeta|^{\lambda-2} F_m(z) \overline{[F(\zeta) - F_m(\zeta)]} \, d\mu(z) \, d\mu(\zeta) \to 0. \end{aligned}$$

Adding these expressions, we find $||F - F_m||^2 \to 0$ as $m \to \infty$. By considering $Q_1 = \{w \;:\; |w| \leq \varepsilon\}$ and $Q_2 = Q_1^c$, we can show as above that $||\hat{F} - \hat{F}_m||^2 \to 0$ (using $\int |\hat{F} - \hat{F}_m|^2 \, d\mu = \int |F - F_m|^2 \, d\mu \to 0$), and so $||F||^2 = ||\hat{F}||^2$.

Now we are ready to consider a general member F of $\mathcal{H}'_\lambda$. For $C > 0$ define

$$F_C(z) = \begin{cases} F(z) & |z| \leq C \\ 0 & \text{else} \end{cases}.$$

Then $||\hat{F}_{C_1} - \hat{F}_C||^2 = ||F_{C_1} - F_C||^2 \to 0$, as $C, C_1 \to \infty$, by the definition of $\mathcal{H}'_\lambda$. Hence the functions $\hat{F}_C$ have a limit in $\mathfrak{H}_\lambda$, which we denote $\hat{F}$, and the equalities $||F_C|| = ||\hat{F}_C||$ imply $||F|| = ||\hat{F}||$. □

Corollary 4.2.3 *For $\lambda \in (0,2)$ we have*

$$\mathcal{H}_\lambda = \left\{ F : \mathbb{C} \xrightarrow{\text{mble}} \mathbb{C} \;\middle|\; \int_{\mathbb{C}} \int_{\mathbb{C}} |z-\zeta|^{\lambda-2} F(z) \overline{F(\zeta)} \, d\mu(z) \, d\mu(\zeta) < \infty \right\}.$$

□

For $n = 0$ the non-compact picture simplifies to

$$(g \cdot F)(z) = |d - bz|^{-2-\lambda} F\left(\frac{az - c}{d - bz}\right). \tag{4.14}$$

Theorem 4.2.4 *The action of $SL(2,\mathbb{C})$ on $\mathcal{H}_\lambda$ given by (4.14), $\lambda \in (0, 2)$, gives an irreducible unitary representation C^λ of $SL(2,\mathbb{C})$. The representations C^λ are called the* complementary series.

Proof. It is easily checked, using Corollary 4.2.3 that the action preserves $\mathcal{H}_\lambda$ and its norm. The proof of irreducibility is along the lines of that in Theorem 4.1.1. □

4.3 The Unitary Dual

Theorem 4.3.1 *The unitary principal series $\pi_{n,i\nu}$ ($n \in \mathbb{N}$, $\nu \in \mathbb{R}$) and the complementary series C^λ ($0 < \lambda < 2$) yield all the non-trivial irreducible unitary representations of $SL(2,\mathbb{C})$. Furthermore, the only equivalences in this list are $\pi_{n,i\nu} \cong \pi_{-n,-i\nu}$.* □

One proof of this follows the same lines as the classification we carried out for $SL(2,\mathbb{R})$. We can first classify the irreducible $(\mathfrak{g}, K)$-modules, then find out which ones are unitarizable, and finally match that list to the members of the unitary principal and complementary series. A reference for this is Taylor [128]. The other approach is by character theory. As for $SL(2,\mathbb{R})$, the irreducible unitary representations have global characters in the sense of distributions on the group. These distributions can be shown to come from locally integrable functions, and this information leads to the result that every unitary representation is a direct integral involving the principal and complementary series. In particular, we get the information that these two series constitute the entire unitary dual, and that the unitary principal series constitutes the reduced dual. The last three sentences sum up the approach of Gelfand and Naimark.

The topology of the unitary dual is natural in the same way as for $SL(2,\mathbb{R})$. Let us first observe what happens to the complementary series at the end-points 0 and 2 of the range of its parameter λ.

At $\lambda = 0$ we see from Lemma 4.2.1 that $\mathfrak{H}_0 = L^2(\mathbb{C})$, and the Fourier transform intertwines the complementary series action on it with the principal series representation $\pi_{0,0}$.

At $\lambda = 2$, (4.7) gives $\langle F, F\rangle = \int_{\mathbb{C}}\int_{\mathbb{C}} F(z)\overline{F(\zeta)}\, d\mu(z)\, d\mu(\zeta) = |\int_{\mathbb{C}} F(z)\, d\mu(z)|^2$, and so the inner product is only positive semi-definite. Consider the surjective map $T : \mathcal{H}' \to \mathbb{C}$, $F \mapsto \int F\, d\mu$. This gives an isomorphism $\mathcal{H}'/\mathcal{H}(0) \to \mathbb{C}$, where $\mathcal{H}(0) = \{F \in \mathcal{H}' \mid \int F\, d\mu = 0\}$ is the kernel of the inner product. Therefore the completion $\mathcal{H}_2$ of $\mathcal{H}'$ is isomorphic to $\mathcal{H}'/\mathcal{H}(0)$. Moreover,

$$\int_{\mathbb{C}} (g \cdot F)(z)\, d\mu(z) = \int_{\mathbb{C}} |d - bz|^{-4} F\left(\frac{az - c}{d - bz}\right) d\mu(z) = \int_{\mathbb{C}} F(\zeta)\, d\mu(\zeta),$$

and hence $\mathcal{H}_2$ is the trivial one dimensional representation.

Further, on the space $\mathcal{H}(0)$ one can put a metric so that its completion is equivalent to the representation $\pi_{2,0}$. Define the operator $D = \partial/\partial\bar{z}$, and the spaces

$$\begin{aligned}\mathcal{E} &= \left\{f \in L^2(\mathbb{C}) \;\middle|\; f \text{ is } C^\infty \text{ except at finitely many points}\right\},\\ \mathcal{F} &= D(\mathcal{E}).\end{aligned}$$

Let π_2 be the action of G on members of $\mathcal{F}$ defined by equation (4.14), with $\lambda = 2$. Then a direct calculation gives the following:

Lemma 4.3.2 *$\pi_{2,0}$ preserves $\mathcal{E}$ and $f \in \mathcal{E}$ implies $D \circ \pi_{2,0}(g)f = \pi_2(g) \circ Df$.* □

Lemma 4.3.3 *D is one-one on $\mathcal{E}$.*

Proof. Suppose $f \in \mathcal{E}$ and $Df = 0$. Then f is an analytic function with finitely many singularities. For simplicity let 0 be a singularity and expand f about it in a Laurent series valid in $B = \{0 < |z| \le R\}$: $f = \sum_{-\infty}^{\infty} a_n z^n$. Put $f_1(z) = \sum_{-\infty}^{-1} a_n z^n$, and let A be the annulus $\{\rho \le |z| \le R\}$, $0 < \rho < R$. Then

$$\begin{aligned}\int_A |f_1|^2\, d\mu &= \sum_{-\infty}^{-1} |a_n|^2 \int_A |z|^{2n}\, d\mu(z)\\ &= 2\pi|a_1|^2 \log\left(\frac{R}{\rho}\right) \sum_{-\infty}^{-2} 2\pi|a_n|^2 \frac{\rho^{2n+2} - R^{2n+2}}{-2-2n}.\end{aligned} \tag{4.15}$$

Now f is L^2 and $f - f_1$ is bounded on B, and so f_1 is L^2 on B. This implies $\lim_{\rho\to 0} \int_A |f_1|^2\, d\mu < \infty$, which contradicts (4.15) unless $a_n = 0$ for every $n < 0$. Thus we see f is an analytic L^2 function, and hence $f = 0$. □

Lemma 4.3.4 *$\mathcal{H}(0) \subset \mathcal{F}$. $D^{-1}\mathcal{H}(0)$ is dense in $L^2(\mathbb{C})$.*

Proof. $\mathcal{H}'$ is obviously contained in $\mathcal{E}$. We have

$$\widehat{Df}(w) = \frac{i}{2} w \hat{f}(w), \tag{4.16}$$

and so $f \in \mathcal{E}$ implies $\widehat{Df}(0) = 0$, hence $D(\mathcal{H}') \subset \mathcal{H}_0$. Since $\mathcal{H}'$ is dense in $L^2(\mathbb{C})$, this implies that $D^{-1}\mathcal{H}(0)$ is dense in $L^2(\mathbb{C})$. Now let $F \in \mathcal{H}(0)$. We want to find $f \in \mathcal{E}$ such that $Df = F$. Combining the requirement $\widehat{Df} = \hat{F}$ with equation (4.16) we arrive at the equation

$$\hat{f}(w) = -\frac{2i}{w}\hat{F}(w).$$

Now $\hat{F}$ is C^∞, rapidly decreasing, and satisfies $\hat{F}(0) = 0$. Hence the function $\psi(w) = -(2i/w)\hat{F}(w)$ is bounded and rapidly decreasing. In particular it is a member of $L^1(\mathbb{C}) \cap L^2(\mathbb{C})$. Therefore it has an inverse Fourier transform and we can define

$$f(z) = \frac{1}{2\pi} \int \psi(w) e^{i\mathrm{Re}\,(\bar{w}z)}\, d\mu(w).$$

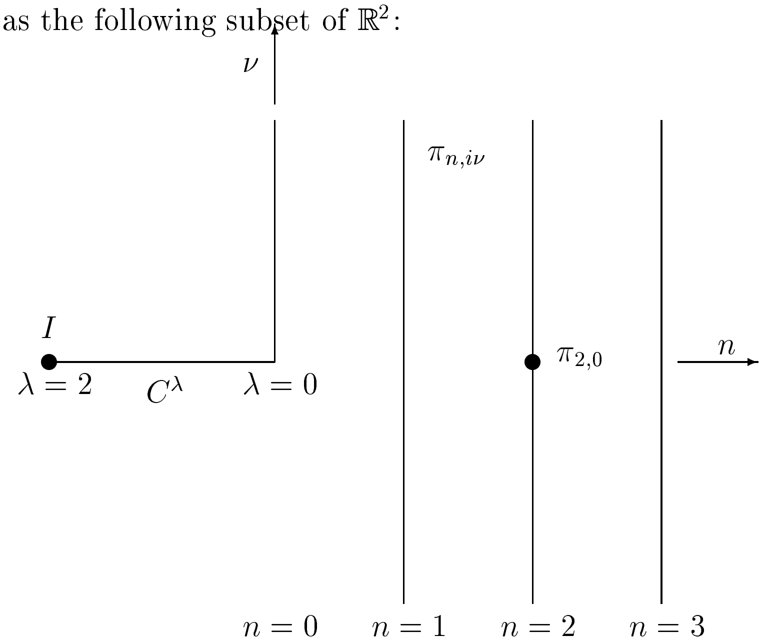
Let us check that this f does the job. Let $m \in \mathcal{H}'$. Then we have

$$\begin{aligned}
\int (Df)\, \bar{m}\, d\mu &= -\int f\, \overline{D'm}\, d\mu && (4.17)\\
&= \frac{i}{2} \int \psi(w)\, w\, \overline{\hat{m}(w)}\, d\mu(w) && (4.18)\\
&= \int \hat{F}(w)\, \overline{\hat{m}(w)}\, d\mu(w) && (4.19)\\
&= \int F\, \bar{m}\, d\mu, && (4.20)
\end{aligned}$$

where $D' = \partial/\partial z$, (4.17) is by integration by parts, (4.18) and (4.20) invoke the Plancherel theorem and the fact $\widehat{D'm}(w) = (i/2)\bar{w}\, \hat{m}(w)$, while (4.19) is by the definition of ψ. The resultant equality implies $Df = F$. $\square$

Thus we can use D to carry the L^2 norm of $\mathcal{E}$ across to $\mathcal{F}$ and then $\mathcal{H}(0)$ is a dense subspace of $\mathcal{F}$. If we take the completions on each side, we can extend D to an isometry of $L^2(\mathbb{C})$ with the completion $\overline{\mathcal{H}(0)}$ of $\mathcal{H}(0)$, and it intertwines the $\pi_{2,0}$ action on $L^2(\mathbb{C})$ with the π_2 action on $\overline{\mathcal{H}(0)}$.

These coincidences are reflected in the topology of the unitary dual of $SL(2,\mathbb{C})$, calculated by Fell [33].[5] By Theorem 4.3.1 the unitary dual can be represented as the following subset of $\mathbb{R}^2$:



The Fell topology of $\widehat{G}$ is the natural topology of this figure as a subset of $\mathbb{R}^2$, with the following modification: As $\lambda \to 2$, the points C^λ converge to both

[5] Fell [34] later determined the structure of $C^*(SL(2,\mathbb{C}))$ itself.

the identity representation I and the principal series representation $\pi_{2,0}$.[6]

From this representation of the unitary dual we see the Fell topology is T_1 and so $SL(2,\mathbb{C})$ is CCR. One could reach the same conclusion by showing the existence of characters (as was done for $SL(2,\mathbb{R})$.

[6] $SL(2,\mathbb{R}$ or $\mathbb{C})$ are something of an exception, for Kazhdan [73] and Wang [146] have shown that for a simple connected Lie group G with finite center and Iwasawa decomposition $G = KAN$, the trivial representation is an isolated point in the unitary dual whenever $\dim_{\mathbb{R}}(A) \geq 2$.