Chapter 2

Real Reductive Lie Groups

The Lorentz groups were the first non-compact simple groups whose representations were successfully classified. The finite dimensional representations were worked out early but it was also known that these are not unitary. Dirac [18] indicated some applications for the unitary infinite dimensional representations, and also constructed some of them. Wigner [152] gave a more complete treatment of the inhomogeneous Lorentz group[1] and showed that it is Type I.

The first complete success was Bargmann's work [6] on $G = SL(2,\mathbb{R})$ published in 1947. He constructed various families of irreducible unitary representations of this group and showed that these constitute all of the unitary dual and their matrix coefficients yield a dense subset of $L^2(G)$. A striking aspect was his proof that a part of the unitary dual (the complementary series) does not contribute to $L^2(G)$, and that another part (the discrete series) occurs as direct summands of $L^2(G)$. Also in 1947, Gelfand and Naimark [38] published their classification of the unitary dual of $SL(2,\mathbb{C})$. They also found the Plancherel formula for this group. $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ became the essential motivation and source for ideas in the subject, and the lines of attack of Bargmann and Gelfand–Naimark spawned the remarkable generalizations of Harish-Chandra to the general class of semisimple Lie groups. In particular, the realizations of the irreducible unitary representations in function spaces were recognized by Mackey as an instance of his method of induction from representations of subgroups.

In the first section of this chapter we describe the class of groups we will be looking at. In the next we indicate some of the basic concepts involved in Harish-Chandra's approach of translating the problems to the Lie algebra setting. In the third we recall basic structural facts about semisimple Lie algebras. Section four focuses on the center of the universal enveloping algebra and the invariant called the infinitesimal character. Finally, Section five sums up the Borel-Weil theory for representations of compact Lie groups.

[1] Also called the *Poincaré group*.

2.1 The Groups Concerned

In this section we describe the groups of our title. The reader needs to be warned that there is no universally agreed upon definition of "real reductive group". The choices made by particular authors depend upon their personal taste and which examples they want to include.

First we recall some definitions and facts in order to fix our notation.

Definition 2.1.1 A *Lie algebra* is a vector space with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, and denoted $(X, Y) \mapsto [X, Y]$, such that:

1. $[X, Y] = -[Y, X]$, $\forall X, Y \in \mathfrak{g}$.
2. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, $\forall X, Y, Z \in \mathfrak{g}$.

Example 2.1.2 The basic example of a Lie algebra is the vector space $L(V)$ of linear operators on a vector space V. The bracket is defined by $[A, B] = A \circ B - B \circ A$. □

Example 2.1.3 If G is a Lie group, then the tangent space at identity has a natural Lie algebra structure, described in Example 1.0.6. □

Definition 2.1.4 A *representation* (π, V) of $\mathfrak{g}$ consists of a vector space V and a Lie algebra homomorphism from $\mathfrak{g}$ into the Lie algebra $L(V)$ of linear operators on V.

The vector space V on which $\mathfrak{g}$ acts should naturally be over the same field as $\mathfrak{g}$. Most commonly, for our cases, $\mathfrak{g}$ will be real while V will be a complex space seen as a real one.

Example 2.1.5 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\exp : \mathfrak{g} \to G$ the exponential map. If $\pi : G \to GL(V)$ is a continuous finite dimensional representation, then it is automatically C^∞. This means that for each $v \in V$ the map $G \to V$, $g \mapsto \pi(g)(v)$, is infinitely differentiable. We can then obtain a Lie algebra representation of $\mathfrak{g}$ in V, also denoted π, by the formula

$$\pi(X)v = \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)v.$$

□

Example 2.1.6 Consider the adjoint action $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Then its derivative map $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g})$, given by $\mathrm{ad}(X)(Y) = [X, Y]$, is a Lie algebra representation, and is called the adjoint representation of $\mathfrak{g}$. □

Some of the definitions regarding group representations (subrepresentation, irreducibility, quotient, direct sum, complex conjugate) apply directly to Lie algebra representations. Others are modified according to the need for Lie algebra actions to correspond to derivatives of Lie group actions.

Definition 2.1.7 Let (π, V) and (ψ, W) be representations of a Lie algebra $\mathfrak{g}$. Their *tensor product representation* $(\pi \otimes \psi, V \otimes W)$ is defined by

$$(\pi \otimes \psi)(X)(v \otimes w) = (\pi(X)v) \otimes w + v \otimes (\psi(X)w).$$

The *contragredient representation* (π^c, V^c) has representation space $V^c = V^*$ and is defined by

$$\pi^c(X) = -\pi(X)^t.$$

Definition 2.1.8 When $\mathfrak{g}$ is a real Lie algebra, we define its *complexification* as follows: The base vector space is $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and the Lie bracket is given by

$$[X \otimes \lambda, Y \otimes \mu] = [X, Y] \otimes \lambda\mu.$$

We denote the complexification by $\mathfrak{g}_{\mathbb{C}}$ (however we shall change our notation from the next section onwards). Then the $\mathbb{R}$-linear representations of $\mathfrak{g}$ in complex vector spaces correspond to $\mathbb{C}$-linear representations of $\mathfrak{g}_{\mathbb{C}}$ in the same vector spaces. Conversely, we call $\mathfrak{g}$ a *real form* of $\mathfrak{g}_{\mathbb{C}}$.

The *Killing form* on a Lie algebra is defined by $(X, Y) = \mathrm{Tr}\,(\mathrm{ad}(X)\mathrm{ad}(Y))$. It is invariant under the adjoint action:

$$([X, Y], Z) + (Y, [X, Z]) = 0 \qquad \forall X, Y, Z \in \mathfrak{g}.$$

Thus, if $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ then $\mathfrak{a}^\perp = \{X \in \mathfrak{g} : \ (X, \mathfrak{a}) = 0\}$ is also an ideal.

Another important object is the *Cartan involution*, which we first define for the general linear group $G = GL(n, \mathbb{C})$ of invertible linear transformations from $\mathbb{C}^n$ to itself. It is the map $\theta : G \to G$ given by $\theta(g) = (g^*)^{-1}$, where g^* denotes the adjoint or conjugate transpose. On the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ we have its differential, which goes by the same name and notation: $\theta(X) = -X^*$. Suppose now that $\mathfrak{g}$ is a real Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ and is closed under adjoint. Then θ restricts to $\mathfrak{g}$. Define $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ by

$$\langle X, Y \rangle = -\mathrm{Re}\,\mathrm{Tr}\,(X\,\theta(Y)).$$

Then $\langle X, X \rangle = \mathrm{Re}\,\mathrm{Tr}\,(XX^*) > 0$ if $X \neq 0$. Hence $\langle \cdot\,, \cdot \rangle$ is an inner product on the real vector space $\mathfrak{g}$. For any linear map $T : \mathfrak{g} \to \mathfrak{g}$ we can now define its *adjoint* T^* by $\langle TX, Y \rangle = \langle X, T^*Y \rangle \ \forall X, Y, Z \in \mathfrak{g}$. Then it is a straight-forward check that

$$\mathrm{ad}(\theta(X)) = -\mathrm{ad}(X)^*,$$

and hence θ is related to the Killing form of $\mathfrak{g}$ by

$$-(X, \theta(X)) = \mathrm{Tr}\,((\mathrm{ad}(X)\mathrm{ad}(X)^*). \tag{2.1}$$

Definition 2.1.9 A Lie algebra is *simple* if it is non-abelian and has no proper non-zero ideal. It is *semisimple* if it is non-abelian and has no non-zero solvable ideal.

Theorem 2.1.10 *A Lie algebra is semisimple if and only if one of the following holds:*

1. *The Killing form is non-degenerate.*
2. *It is a Lie algebra direct sum of simple ideals.*
3. *Every finite dimensional representation is completely reducible.* □

A real Lie algebra is semisimple if and only if its complexification is semisimple. However the complexification of a simple Lie algebra need not be simple. For example, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is simple when seen as a real Lie algebra, but its complexification is $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ which is not simple.

Definition 2.1.11 A Lie group is *semisimple* if its Lie algebra is semisimple.

Theorem 2.1.12 *Let G be a closed subgroup of $GL(n,\mathbb{C})$ such that it is closed under adjoint and has a finite center. Then its Lie algebra is semisimple.*

Proof. Since the center of G is finite, the center of its Lie algebra $\mathfrak{g}$ is 0. Hence if X is a non-zero element of $\mathfrak{g}$ then $\mathrm{ad}(X)$ is also non-zero, and equation (2.1) implies that $(X, \theta(X)) < 0$. Therefore the Killing form of $\mathfrak{g}$ is non-degenerate and $\mathfrak{g}$ is semisimple. □

We now list a class of examples of semisimple groups. To start with, we have the *complex classical groups*:

$$\begin{aligned} SL(n,\mathbb{C}) &= \{g \in GL(n,\mathbb{C}) :\ \det g = 1\}, \\ SO(n,\mathbb{C}) &= \{g \in SL(n,\mathbb{C}) :\ g^t g = I_n\}, \\ Sp(n,\mathbb{C}) &= \{g \in SL(2n,\mathbb{C}) :\ g^t J g = J\}. \end{aligned}$$

Here, I_n is the $n \times n$ identity matrix, and $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Our next list is of the *non-compact real classical groups*:

$$\begin{aligned} SL(n,\mathbb{R}) &= \{g \in GL(n,\mathbb{R}) :\ \det g = 1\}, \\ SL(n,\mathbb{H}) &= \{g \in GL(n,\mathbb{H}) :\ \det g = 1\}, \\ SO(m,n) &= \{g \in SL(m+n) :\ g^t I_{m,n} g = I_{m,n}\}, \\ SU(m,n) &= \{g \in SL(m+n,\mathbb{C}) :\ g^* I_{m,n} g = I_{m,n}\}, \\ Sp(m,n) &= \{g \in Sp(m+n,\mathbb{C}) :\ g^t Q \bar{g} = Q\}, \\ Sp(n,\mathbb{R}) &= Sp(n,\mathbb{C}) \cap SL(2n,\mathbb{R}) \ \cong\ \{g \in SU(n,n) : g^t J g = J\}, \\ SO^*(2n) &= \{g \in SU(n,n) : g^t \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right) g = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)\}. \end{aligned}$$

In this list,[2] J is as before, $I_{m,n} = \left(\begin{smallmatrix} I_m & 0 \\ 0 & -I_n \end{smallmatrix}\right)$, and $Q = \left(\begin{smallmatrix} I_{m,n} & 0 \\ 0 & I_{m,n} \end{smallmatrix}\right)$.

One usually seeks to piece together the representation theory of a group from that of certain subgroups, using Mackey induction. As we shall see, in the case of a semisimple group the subgroups that are important are not quite semisimple.

Example 2.1.13 For the special linear group $SL(n,\mathbb{R})$ the "certain subgroups" alluded to above are the block diagonal ones. For instance when we study the representations of $G = SL(3,\mathbb{R})$ we encounter the following group:

$$L = \left\{ \begin{pmatrix} X & 0 \\ 0 & t \end{pmatrix} \in SL(3,\mathbb{R}) \;\middle|\; X \in GL(2,\mathbb{R}), t \in \mathbb{R}^\times \right\}.$$

[2] $\mathbb{H}$ is the quaternion algebra. The identification $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ gives a semisimple group structure to $SL(n,\mathbb{H})$.

Note that G is connected but L is disconnected. The Lie algebra of L is given by

$$\mathfrak{l} = \{X \oplus s \in \mathfrak{gl}(2,\mathbb{R}) \oplus \mathbb{R} \mid \mathrm{Tr}\, X + s = 0\}.$$

It is easy to calculate that the center of $\mathfrak{l}$ is $\mathfrak{z} = \{\lambda I \oplus s \mid 2\lambda + s = 0\}$ and that $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}$. Further, $[\mathfrak{l},\mathfrak{l}] = \mathfrak{sl}(2,\mathbb{R})$ is semisimple. □

Therefore we broaden our discussion to the type of group defined below.

Definition 2.1.14 Let $\mathfrak{g}$ be a Lie algebra with center $\mathfrak{z}(\mathfrak{g})$. Then $\mathfrak{g}$ is *reductive* if $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$ with $[\mathfrak{g},\mathfrak{g}]$ semisimple.

The following result can be found in most texts on Lie groups.[3]

Theorem 2.1.15 *A Lie algebra $\mathfrak{g}$ is reductive if and only if one of the following conditions holds:*

1. *The adjoint action of $\mathfrak{g}$ on itself is completely reducible.*
2. *There is a faithful completely reducible finite dimensional representation of $\mathfrak{g}$.* □

Corollary 2.1.16 *Let K be a compact Lie group. Then its Lie algebra is reductive.* □

Now we introduce the *compact classical groups*:

$$\begin{array}{rcl} O(n) & = & \{g \in GL(n,\mathbb{R}) : \ g^t g = I_n\}, \\ SO(n) & = & \{g \in SL(n,\mathbb{R}) : \ g^t g = I_n\}, \\ U(n) & = & \{g \in GL(n,\mathbb{C}) : \ g^* g = I_n\}, \\ SU(n) & = & \{g \in U(n) : \det g = 1\}, \\ Sp(n) & = & \{g \in U(2n) : \ g^t J g = J\}. \end{array}$$

Theorem 2.1.17 *Let G be a closed subgroup of $GL(n,\mathbb{C})$ which is closed under adjoint. Then its Lie algebra is reductive.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{z}$ be its center. For any $X \in \mathfrak{g}$ let $\overline{X}$ be the corresponding coset in $\mathfrak{g}/\mathfrak{z}$. If $T : \mathfrak{g} \to \mathfrak{g}$ is a linear map such that $T|_{\mathfrak{z}} \equiv 0$ then it induces a linear map $T' : \mathfrak{g}/\mathfrak{z} \to \mathfrak{g}/\mathfrak{z}$ by $T'(\overline{X}) = \overline{TX}$. Then $\mathrm{Tr}\, T' = \mathrm{Tr}\, T$. Further, if $S : \mathfrak{g} \to \mathfrak{g}$ is another such map, then $S'T' = (ST)'$. Now let $\overline{X}$ be a non-zero element in $\mathfrak{g}/\mathfrak{z}$. Then we have

$$(\overline{X}, \overline{\theta(X)}) = \mathrm{Tr}\,(\mathrm{ad}(X)'\mathrm{ad}(\theta(X))') = \mathrm{Tr}\,(\mathrm{ad}(X)\mathrm{ad}(\theta(X))) = (X, \theta(X)),$$

and this is non-zero by equation (2.1) since $\mathrm{ad}(X) \neq 0$. Therefore $\mathfrak{g}/\mathfrak{z}$ is semisimple. On $\mathfrak{g}$ we put the non-degenerate real-valued form $\langle\langle X, Y\rangle\rangle = \mathrm{Re}\,\mathrm{Tr}\,(XY)$, and let $\mathfrak{z}^\perp$ be the corresponding orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$. Now $\mathfrak{z}$ is a θ-stable ideal and so the restriction of $\langle\langle \cdot, \cdot\rangle\rangle$ to $\mathfrak{z}$ is non-degenerate (consider $\langle\langle Z, \theta(Z)\rangle\rangle$). Hence $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ and $[\mathfrak{z},\mathfrak{z}] = \mathfrak{z}^\perp \cong \mathfrak{g}/\mathfrak{z}$ is semisimple. □

[3] For instance Varadarajan [132], Theorem 3.16.3.

The definition of a real reductive Lie group G starts with the requirement that its Lie algebra be reductive. Additional hypotheses are added to ease the analysis on the group, and these may vary with context and author. The *Harish-Chandra class*, defined below, suffices for studying semisimple Lie groups (with finite center) by reduction to their Levi subgroups.[4]

Definition 2.1.18 A real Lie group G with Lie algebra $\mathfrak{g}$ is in the *Harish-Chandra class* if:

1. $\mathfrak{g}$ is reductive.
2. G has finitely many connected components.
3. $\mathrm{Ad}(G) \subset \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$, the inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$.
4. The analytic subgroup G_1 of G, corresponding to $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ has finite center.

The Harish-Chandra class contains all the classical non-compact groups mentioned so far, as well as $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$.[5] But it does not contain $O(n)$ as that group does not satisfy condition 3 of the definition.

Note that the Cartan involution $\theta : GL(n,\mathbb{C}) \to GL(n,\mathbb{C})$ has as its fixed point set the group $U(n)$ which is a maximal compact subgroup. At the Lie algebra level we have the corresponding *Cartan decomposition* $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}(n)$ where $\mathfrak{u}(n)$ = skew Hermitian matrices and $\mathfrak{p}(n)$ = Hermitian matrices, are the $+1$ and -1 eigenspaces of θ. These observations generalize to all the groups covered by Theorem 2.1.10, as θ restricts to them and we can list the maximal compact subgroup K of each non-compact classical group G:

G	K	G	K
$GL(n,\mathbb{C})$	$U(n)$	$SL(n,\mathbb{C})$	$SU(n)$
$SO_0(n,\mathbb{C})$	$SO(n)$	$SL(n,\mathbb{R})$	$SO(n)$
$SL(n,\mathbb{H})$	$Sp(n)$	$SO_0(m,n)$	$SO(m)\times SO(n)$
$SU(m,n)$	$S(U(m)\times U(n))$	$Sp(m,n)$	$Sp(m)\times Sp(n)$
$Sp(n,\mathbb{R})$	$U(n)$	$SO^*(2n)$	$U(n)$

This motivates the following definition:

Definition 2.1.19 A *Cartan involution* is a Lie group homomorphism $\theta : G \to G$ such that $\theta^2 = 1$ and $K = \{g \in G |\ \theta(g) = g\}$ is a maximal compact subgroup of G.

Wallach [144] has a more involved definition, which has the virtue that the properties of the groups are put in plain view. Let $G_{\mathbb{C}}$ be a subgroup of $GL(n,\mathbb{C})$ defined as the common zeros of a finite set of complex polynomials on the matrix algebra $M(n,\mathbb{C})$, and closed under adjoint. $G_{\mathbb{R}} = G_{\mathbb{C}} \cap GL(n,\mathbb{R})$ is the group of *real points.*

[4]Levi subgroups are described in Chapter 5. The group L of Example 2.1.5 is an instance.

[5]This class of groups was introduced by Harish-Chandra in [53]. A detailed discussion of the structure of its members can be found in Varadarajan [131, Part II, Section 1].

Definition 2.1.20 G is a *real reductive group* if it is a finite cover of an open subgroup of $G_{\mathbb{R}}$.

All the non-compact classical groups exist as reductive groups of type $G_{\mathbb{R}}$. A general connected semisimple Lie group G with finite center, on the other hand, is a finite cover of the identity component of $G_{\mathbb{R}} = \mathrm{Aut}(\mathfrak{g})$, where $G_{\mathbb{C}} = \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. From the definition it is clear that the Lie algebra of a real reductive G is reductive, since it is identified with that of $G_{\mathbb{R}}$ and Theorem 2.1.16 applies to $G_{\mathbb{R}}$.

A real reductive group is of *inner type* if $\mathrm{Ad}(G)$ is a subgroup of $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. With this additional hypothesis, Wallach's definition becomes the same as that of the Harish-Chandra class. Hereafter, **whenever we refer to a general non-compact real reductive group, we will assume it is of inner type.** [6]

Theorem 2.1.21 *Let G be a real reductive group as in 2.1.12. Then it has an Iwasawa decomposition $G = KAN$, where K is a maximal compact subgroup of G, A is abelian, and N is nilpotent.* □

The Iwasawa decomposition is described in detail in Section 5.1. The requirement that G_1 have a finite center (or, in Wallach's definition, that G be a finite cover) is imposed to have a compact K in this decomposition.

An easy consequence of our definitions is the following:

Theorem 2.1.22 *Real reductive Lie groups are unimodular.*

Proof. Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$. Then $\triangle_G(e^X) = |\det \mathrm{Ad}(e^X)| = |\det e^{\mathrm{ad}(X)}| = e^{\mathrm{Tr}\,\mathrm{ad}(X)} = 1$, since $\mathrm{Tr}\,\mathrm{ad}(X) = 0$ for reductive Lie algebras. Therefore $\triangle_G(e^X) \equiv 1$ if G is connected. In general, G has finitely many components, and then $\triangle_G$ is a homomorphism onto a finite subgroup of the multiplicative group $\mathbb{R}^\times$. But the only such subgroup is $\{1\}$. □

2.2 Harish-Chandra Modules

Harish-Chandra was born in Kanpur in 1923. His scientific career started off in Physics — he got his B.Sc. and M.Sc. from Allahabad University in that subject. He then went to I.I.Sc. Bangalore as a student of Bhabha. Two years later he went to Cambridge to work with Dirac, and got his Ph.D. in 1947. Dirac went to the Institute for Advanced Study, Princeton, that year and Harish-Chandra went with him. There he began his transition from physicist to pure mathematician.

On Dirac's advice he started work on the infinite dimensional representations of the Lorentz group. Along with $SL(2,\mathbb{R})$ this is an example of the class of semisimple Lie groups. Harish-Chandra appears to have realized early that

[6] A discussion of possible meanings of "reductive group" can be found in Wolf [154, §0.3]. The least that is usually required from a choice of definition is that it include the semisimple groups with finite centers, and be hereditary under passage to Levi subgroups.

a study of the structure of semisimple groups, particularly of the invariant differential operators on them, would be the key to extension of Bargmann's results. This went against the prevailing belief of the time that it was the integral operators that would be "the battering ram that would break down the doors and open the vistas to the promised land." [7]

In 1950 he joined Columbia University. Finally, in 1963 he went to the IAS as a permanent member and stayed there till his death in 1983. In these thirty five or so years he took on the hardest problems available in the representation theory of semisimple groups, and in the process invented the modern form of the subject. Accounts of his life and work may be found in [95, 134].

We now launch into a brief survey of some results of Harish-Chandra which form an essential part of the background. Most of them are in his article [46] of 1953. A detailed account is present in Chapter four of Warner [148].

Notation Let G be a real Lie group. The Lie algebra of G is then denoted by $\mathfrak{g}_o$ and its complexification by $\mathfrak{g}$. In the rest of this book we will adopt this convention of using the $_o$ subscript for the Lie algebra of a real Lie group, and dropping it to denote the complexification.

Let $(\pi, \mathcal{H})$ be a continuous representation of G in a separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. Our first desire is to shift from the group action to the Lie algebra action, the algebra being easier to study. This requires the group action on the vectors to be differentiable, but it is not *a priori* obvious that there *any* vectors on which the group acts differentiably. (Recall however that a continuous *finite dimensional* representation of a Lie group is always differentiable.) First, we show that the subspace on which the group has a C^∞ action is, in fact, dense in $\mathcal{H}$.

Definition 2.2.1 Let $v \in \mathcal{H}$. Then $v \in \mathcal{H}^\infty$ if the map $G \to \mathcal{H}$: $g \mapsto \pi(g)v$ is C^∞. Members of $\mathcal{H}^\infty$ are called *smooth* vectors.

Let us add some detail to this definition. A map f from an open subset U of $\mathbb{R}^n$ into a Hilbert space $\mathcal{H}$ is said to be differentiable at $x \in U$ if there exists a linear map $Df_x : \mathbb{R}^n \to \mathcal{H}$ such that

$$\lim_{t \to 0} \frac{f(x+t) - f(x) - Df_x(t)}{||t||} = 0.$$

The map Df_x is called the *derivative* of f at x. If the map $x \mapsto Df_x$ is continuous, we say f is continuously differentiable or C^1. By repeating this process we arrive at the definition of a C^∞ function.

A related notion is of *weak differentiability*: f is weakly C^∞ if for every $v \in \mathcal{H}$, the map $\langle f(\cdot), v \rangle$ from U into $\mathbb{C}$ is C^∞ in the usual sense. It is not hard to see that f is C^∞ if and only if it is weakly C^∞.[8]

[7] Harish-Chandra to V.S. Varadarajan. One notes that integral operators were the key to the Peter-Weyl theorem.

[8] Grothendieck [44] showed that the two notions coincide for maps from $\mathbb{R}^n$ into any Fréchet space.

Theorem 2.2.2 (Gårding) *Let $f \in C_c^\infty(G)$, and $v \in \mathcal{H}$. Then $\pi(f)v = \int_G f(x)\pi(x)v\,dx$ is in $\mathcal{H}^\infty$ and the span of all such vectors is dense in $\mathcal{H}$.*[9]

Proof. $\pi(y)\pi(f)v = \int_G f(x)\pi(yx)vdx = \int_G f(y^{-1}x)\pi(x)vdx$ and one can differentiate with respect to y under the integral sign. To show such vectors are dense, we take an approximate identity in $C_c^\infty(G)$: a sequence $\{f_n\}$ such that $f_n \in C_c^\infty(G)$, $\int_G f_n dx = 1$, and $\mathrm{supp}(f_n) \downarrow \{1\}$ as $n \to \infty$.

Now define $K_n = \mathrm{supp}(f_n)$ and $M_n = \sup_{x \in K_n} ||\pi(x)v - v||$. Then $M_n \to 0$ as $n \to \infty$. Hence we get

$$\begin{aligned} ||\pi(f_n)v - v|| &= ||\int_G f_n(x)\pi(x)v\,dx - \int_G f_n(x)v\,dx|| \\ &= ||\int_{K_n} f_n(x)[\pi(x)v - v]\,dx|| \\ &\leq M_n \to 0 \text{ as } n \to \infty. \end{aligned}$$

□

On $\mathcal{H}^\infty$ we can define a $\mathfrak{g}_o$ action as usual:

$$\pi(X)v = \frac{d}{dt}\Big|_{t=0} \pi(e^{tX})v, \quad X \in \mathfrak{g}_o, \; v \in \mathcal{H}^\infty,$$

and this extends to a $\mathfrak{g}$ action. The G and $\mathfrak{g}$ actions are related by

$$\pi(x)\pi(X)\pi(x)^{-1} = \pi(\mathrm{Ad}(x)X), \tag{2.2}$$

the proof being an application of the equality $e^{\mathrm{Ad}(x)X} = xe^X x^{-1}$ inside G. $\mathcal{H}^\infty$ is both G and $\mathfrak{g}$ invariant.

The problem of globalizing a Lie algebra action to the group is akin to reconstructing a function from its Taylor Series. Therefore, it is not enough to restrict to the smooth vectors. For one thing, the closure of a $\mathfrak{g}$ invariant subspace of $\mathcal{H}^\infty$ is usually not G invariant. This is an obstruction to reducing the study of continuous representations of G to representations of $\mathfrak{g}$.

Example 2.2.3 Consider the left regular representation L of $\mathbb{R}$ on $\mathcal{H} = L^2(\mathbb{R})$. Then it is easy to check that $C_c^\infty(\mathbb{R}) \subset \mathcal{H}^\infty$, and the Gårding subspace is $C_c^\infty(\mathbb{R}) * L^2(\mathbb{R})$. The corresponding action of the Lie algebra is given by $t \mapsto -t\,d/dx$. Now the subspace of smooth functions supported on $[0,1]$ is invariant under the Lie algebra action but its closure is not invariant under the group action. □

Definition 2.2.4 A vector $v \in \mathcal{H}$ is *analytic* under π if the map $g \mapsto \pi(g)v$ is real analytic. $\mathcal{H}^\omega$ denotes the space of analytic vectors.

The analyticity of a vector v is equivalent to its weak analyticity: i.e. the analyticity of all the functions $g \mapsto \langle \pi(g)v, w\rangle$.[10]

---

[9]The span of $\{\pi(f)v : v \in \mathcal{H},\; f \in C_c^\infty(G)\}$ is called the Gårding subspace. In 1978, Dixmier and Malliavin [22] showed that for any Lie group G and representation $\mathcal{H}$ as above, the Gårding subspace equals the full space of smooth vectors.

[10]The equivalence of these notions holds in Banach spaces, but not in general Fréchet spaces. See Barros-Neto [7] for details.

Theorem 2.2.5 *The space $\mathcal{H}^\omega$ of analytic vectors is dense in $\mathcal{H}$.*

Proof. This was proven by Harish-Chandra for semisimple Lie groups. A general proof for all Banach space representations of arbitrary Lie groups was later given by Nelson [109]. □

$\mathcal{H}^\omega$ is also G and $\mathfrak{g}$ invariant. The following result formalizes the transition from G to $\mathfrak{g}$:

Theorem 2.2.6 *Let $V \subset \mathcal{H}^\omega$ be $\mathfrak{g}$-stable. Then its closure* cl$V$ *is G-stable.*

Proof. Let $v \in V$, $w \in V^\perp$, and $X \in \mathfrak{g}_o$. Then for small t we have

$$\langle \pi(e^{tX})v, w\rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!}\langle \pi(X)^n u, w\rangle = 0$$

since V is $\mathfrak{g}$-stable. By the analyticity of v we get

$$\langle \pi(e^{tX})v, w\rangle = 0, \quad \forall t \in \mathbb{R}.$$

Therefore $\exp(\mathfrak{g}_o)$ maps V into $V^{\perp\perp} = \mathrm{cl}V$. Since $\exp(\mathfrak{g}_o)$ generates G, G also maps V into cl$V$. The continuity of π now implies G maps cl$V$ into cl$V$. □

Although we now have a better relationship between the closed G invariant subspaces of $\mathcal{H}$ and the $\mathfrak{g}$ invariant subspaces of $\mathcal{H}^\omega$, $\mathcal{H}^\omega$ is too large. The problem is that it can be reducible under the $\mathfrak{g}$ action even when $\mathcal{H}$ is irreducible under the G action. One further restriction is needed. Let K be a compact subgroup of G. Note that we may, and hence do, assume that the restriction of the action to K is unitary. By the *K-isotype* $\mathcal{H}(\delta)$ we mean the union of all the copies in $\mathcal{H}$ of $\delta \in \widehat{K}$.

Definition 2.2.7 A vector $v \in \mathcal{H}$ is *K-finite* if the span of its K-orbit is finite dimensional. We denote the subspace of K-finite vectors by $\mathcal{H}_K$.

It is easy to see that $\mathcal{H}_K$ is the algebraic direct sum of all the isotypes $\mathcal{H}(\delta)$ and is K invariant. Now let $\mathcal{H}_K^\omega = \sum_{\delta\in\widehat{K}} \mathcal{H}^\omega(\delta)$ be the space of analytic K-finite vectors, where $\mathcal{H}^\omega(\delta) = \mathcal{H}^\omega \cap \mathcal{H}(\delta)$. $\mathcal{H}_K^\omega$ is called the underlying *Harish-Chandra module* of $\mathcal{H}$.

Theorem 2.2.8 *$\mathcal{H}_K^\omega$ is dense in $\mathcal{H}$.*

Proof. Let $u \in \mathcal{H}$ and $\epsilon > 0$. By Theorem 2.2.5 there is an analytic vector v such that

$$||u - v|| < \epsilon/2.$$

Now, for $\delta \in \widehat{K}$, recall that

$$P(\delta)w = \int_K \overline{\chi_\delta(k)}\pi(k)(w)\, dk$$

gives the canonical projection of $\mathcal{H}$ onto $\mathcal{H}(\delta)$ (Theorem 1.1.5). By the theory for compact groups, $\mathcal{H}$ is the Hilbert space direct sum of the $\mathcal{H}(\delta)$'s, and there is a finite $F \subset \widehat{K}$ such that

$$||v - \sum_{\delta\in F} P(\delta)v|| < \epsilon/2.$$

Since $v \in \mathcal{H}^\omega$ and $\chi_\delta \in C^\omega(K)$, we also have $P(\delta)v \in \mathcal{H}^\omega$, and consequently $\sum_{\delta\in F} P(\delta)v \in \sum_{\delta\in\widehat{K}} \mathcal{H}^\omega(\delta) = \mathcal{H}^\omega_K$. Thus we have found an element of $\mathcal{H}^\omega_K$ which is within ϵ of u. □

$\mathcal{H}^\omega_K$ is, in general, not stable under G. However, we have:

Theorem 2.2.9 *$\mathcal{H}^\omega_K$ is invariant under the $\mathfrak{g}$ and K actions.*

Proof. Invariance under the K action is clear. As for $\mathfrak{g}$ invariance, the relation (2.2) gives:

$$\pi(k)\pi(X)v = \pi(\mathrm{Ad}(k)X)\pi(k)v \quad \forall v \in \mathcal{H}^\omega,\ k \in K,\ X \in \mathfrak{g}_o.$$

In particular, when $v \in \mathcal{H}^\omega_K$, the vectors $\pi(k)v$ lie in a finite dimensional space (as k varies), while $\pi(\mathrm{Ad}(k)X)$ lies in $\pi(\mathfrak{g}_o)$ which is also finite dimensional. Consequently, $\pi(X)v \in \mathcal{H}^\omega_K$. □

This collection of results determines our basic definition.

Definition 2.2.10 Let V be a complex vector space, acted on by $\mathfrak{g}$ and K. V is called a $(\mathfrak{g}, K)$ *module* if:

1. The differential of the K action equals the action of $\mathfrak{k}_o$ as a subalgebra of $\mathfrak{g}$.
2. $\pi(Ad(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$ for all $X \in \mathfrak{g},\ k \in K$.
3. The span of the K orbit of each $v \in V$ is finite dimensional, and the action of K on this space is continuous with respect to the natural topology on it. In particular, $V = \sum_{\delta\in\widehat{K}} V(\delta)$, as an algebraic direct sum.

A subspace W of a $(\mathfrak{g}, K)$ module V is called a $(\mathfrak{g}, K)$ submodule if it is invariant under the $\mathfrak{g}$ and K actions. V is called *irreducible* if its only $(\mathfrak{g}, K)$ submodules are itself and $\{0\}$.

By our choice of definition, if G has a continuous representation in a separable Hilbert space $\mathcal{H}$, then $\mathcal{H}^\omega_K$ is a $(\mathfrak{g}, K)$ module.

Definition 2.2.11 Two representations $\mathcal{H}$ and $\mathcal{H}'$ are said to be *infinitesimally equivalent* if $\mathcal{H}^\omega_K$ and $\mathcal{H}'^\omega_K$ are isomorphic as $(\mathfrak{g}, K)$ modules.

An important subcollection of $(\mathfrak{g}, K)$ modules is given below:

Definition 2.2.12 A $(\mathfrak{g}, K)$ module V is *admissible* if each K-isotype $V(\delta)$ is finite dimensional. Correspondingly, a G representation in $\mathcal{H}$ is called admissible if each $\mathcal{H}(\delta)$ is finite dimensional.

Since $\mathcal{H}^\omega(\delta)$ is dense in $\mathcal{H}(\delta)$, the finite dimensionality of one is equivalent to that of the other. In particular, the admissibility of $\mathcal{H}$ is equivalent to that of $\mathcal{H}^\omega_K$. Another easy consequence is that $\mathcal{H}_K = \mathcal{H}^\omega_K$ when $\mathcal{H}$ is admissible. Within the context of admissible representations, the transition to an algebraic setting is complete:

Theorem 2.2.13 *Let $\mathcal{H}$ be an admissible representation of G. Then the $(\mathfrak{g}, K)$ submodules V of $\mathcal{H}_K$ are in bijection with the closed G invariant subspaces $\mathcal{V}$ of $\mathcal{H}$. The correspondence is given in one direction by $V \mapsto \mathrm{cl}V$, and in the other by taking the (analytic) K-finite vectors of $\mathcal{V}$.* □

Our ultimate aim is an understanding of the irreducible unitary representations of G. Now we relate the behaviour of unitarity under this procedure of passing to the underlying Harish-Chandra modules.

Definition 2.2.14 A $(\mathfrak{g}, K)$ module V is called *unitarizable* if there is a positive definite inner product $\langle\cdot,\cdot\rangle$ on V such that:

1. $\langle kv, kw\rangle = \langle v, w\rangle,\ k \in K$,
2. $\langle Xv, w\rangle = -\langle v, Xw\rangle,\ X \in \mathfrak{g}_o$.

If $\mathcal{H}$ is a unitary representation of G, then $\mathcal{H}_K^\omega$ is clearly a unitarizable $(\mathfrak{g}, K)$ module.

Let us now restrict to the case where G is a real reductive Lie group and K is a maximal compact subgroup. The next result shows that the category of admissible modules is large enough to be useful.

Theorem 2.2.15 *Irreducible $(\mathfrak{g}, K)$ modules are admissible.* □

While at the global level we have:

Theorem 2.2.16 *Irreducible unitary representations of G are admissible.* □

Further:

Theorem 2.2.17 *If $(\pi_i, \mathcal{H}^i)$, $i = 1, 2$, are unitary, admissible representations of G then π_1,π_2 are unitarily equivalent if and only if they are infinitesimally equivalent.* □

For reductive groups, the Subquotient Theorem realizes every irreducible $(\mathfrak{g}, K)$ module V as the underlying Harish-Chandra module of some Hilbert space representation of G. If V is unitarizable then $\mathcal{H}$ can be chosen to be unitary. In Chapter 3, for our initial example of $SL(2, \mathbb{R})$, we do not assume this theorem but illustrate it by first classifying the irreducible $(\mathfrak{g}, K)$ modules and then explicitly finding their Hilbert space "globalizations". The Subquotient Theorem was first proved by Harish-Chandra [47] for linear semisimple Lie groups, and extended to general connected semisimple Lie groups (with finite center) by Lepowsky [96] and Rader [117]. We discuss it briefly in §7.1.

2.3 Semisimple Lie Algebras

In this section we collect some common knowledge about the universal enveloping algebra and semisimple Lie algebras.[11] The aim is to collect just enough so that in the next section we can prove the Harish-Chandra isomorphism theorem for the center of the universal enveloping algebra. As usual, $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$.

Think of $\mathfrak{g}$ as a vector space and let $\mathcal{T} = \mathcal{T}(\mathfrak{g})$ be its tensor algebra. $\mathcal{I}$ is the two sided ideal generated by $\{X \otimes Y - Y \otimes X - [X, Y] :\ X, Y \in \mathfrak{g}\}$. Then the *universal enveloping algebra* of $\mathfrak{g}$ is defined to be $\mathcal{U}(\mathfrak{g}) = \mathcal{T}/\mathcal{I}$. Let

[11] Good references for this material are [36, 68, 69].

$i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the map given by the sequence $\mathfrak{g} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{I}$. $\mathcal{U}(\mathfrak{g})$ is an associative algebra with 1, and has the following universal property: Let $\mathcal{A}$ be any associative algebra over $\mathbb{C}$, with 1. Denote by $\mathcal{A}_L$ the Lie algebra with underlying vector space $\mathcal{A}$ and Lie bracket given by $[a,b] = ab - ba$. Then, given any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathcal{A}_L$, there is a unique algebra homomorphism $\tilde{\varphi} : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ such that $\tilde{\varphi} \circ i = \varphi$.

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}) \arrow[dr, "\tilde{\varphi}"] & \\
\mathfrak{g} \arrow[u, "i"] \arrow[r, "\varphi"] & \mathcal{A} = \mathcal{A}_L
\end{tikzcd}$$

Thus, Lie algebra representations of $\mathfrak{g}$ correspond to algebra representations of $\mathcal{U}(\mathfrak{g})$.

Theorem 2.3.1 (Poincaré-Birkhoff-Witt) *Suppose $X_1, \ldots, X_m$ is a basis of $\mathfrak{g}$. Then the monomials $X_1^{n_1} \cdots X_m^{n_m}$, $n_i \in \mathbb{Z}_{\geq 0}$, form a basis of $\mathcal{U}(\mathfrak{g})$.* □

It is an immediate consequence that the map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is an embedding. Hence the correspondence between Lie algebra representations of $\mathfrak{g}$ and associative algebra representations of $\mathcal{U}(\mathfrak{g})$ is given by extensions in one direction, and restrictions in the other.

Definition 2.3.2 The *transpose map* from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g})$ is the unique anti-automorphism of $\mathcal{U}(\mathfrak{g})$, $u \mapsto u^t$, such that $X \in \mathfrak{g}$ implies $X^t = -X$.

Now let $\mathcal{S}(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. Consider the map $sym : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ defined by

$$sym(X_1 \cdots X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \cdots X_{\sigma(k)}.$$

The Poincaré-Birkhoff-Witt Theorem can be interpreted in the following language.

Theorem 2.3.3 *$sym : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is a linear isomorphism.* □

Although sym is not an algebra isomorphism in general, it does satisfy the following relation: If $P, Q \in \mathcal{S}(\mathfrak{g})$ are monomials of degree p, q, then $sym(PQ) - sym(P)sym(Q)$ has degree $< p + q$. So sym preserves the term of highest degree.

Definition 2.3.4 A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which equals its own normalizer.

Theorem 2.3.5 *A Cartan subalgebra of a semisimple complex Lie algebra is a maximal abelian subalgebra consisting of elements whose adjoint action is semisimple.* □

From here on $\mathfrak{g}$ is assumed to be a semisimple complex Lie algebra.

Consider the characteristic polynomial of $\mathrm{ad}(X)$, $X \in \mathfrak{g}$. We write

$$\det(tI - \mathrm{ad}(X)) = \sum_{j=0}^{n} t^j D_j(X),$$

where D_j is a polynomial on $\mathfrak{g}$ and $n = \dim \mathfrak{g}$. Let r be the smallest index such that D_r is not identically zero. We say X is *regular* if $D_r(X) \neq 0$. Let $\mathfrak{g}'$ denote the set of regular elements. It is dense in $\mathfrak{g}$.

Theorem 2.3.6 *Let $X \in \mathfrak{g}'$. Then* $\mathrm{ad}(X)$ *is semisimple, and the centralizer in $\mathfrak{g}$ of X is a Cartan subalgebra. Further, every Cartan subalgebra is of this form, for some $X \in \mathfrak{g}'$.* □

For a given Cartan subalgebra $\mathfrak{h}$, there is a decomposition of $\mathfrak{g}$ into common eigenspaces of the elements of $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \triangle} \mathfrak{g}_\alpha,$$

where $\triangle = \triangle(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$, $X \in \mathfrak{g}_\alpha$ if $\mathrm{ad}(H)X = \alpha(H)X \ \forall H \in \mathfrak{h}$, and $\alpha \in \triangle$ if and only if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. If $\alpha \in \triangle$, then it is called a *root* and $\mathfrak{g}_\alpha$ is called a *root space*. Further, $\mathfrak{h} = \mathfrak{g}_0$.

The root system $\triangle$ has subsets called *bases* distinguished by the following property: Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ be a base. Then any $\alpha \in \triangle$ satisfies $\alpha = \sum_{i=1}^n n_i\alpha_i$, $n_i \in \mathbb{Z}$, with all n_i's non-negative or all n_i's non-positive. Thus, for a fixed base, we have a natural notion of *positive* and *negative* roots, denoted by $\triangle^\pm$. Further, $\alpha \in \triangle$ if and only if $-\alpha \in \triangle$, so that $\triangle^- = -\triangle^+$. The members of a base are called *simple* roots.

For a given choice of $\triangle^\pm$ we define the *Borel subalgebras* $\mathfrak{b}^+$ and $\mathfrak{b}^-$ by

$$\mathfrak{b}^+ = \mathfrak{h} \oplus \sum_{\alpha \in \triangle^+} \mathfrak{g}_\alpha, \qquad \mathfrak{b}^- = \mathfrak{h} \oplus \sum_{\alpha \in \triangle^-} \mathfrak{g}_\alpha.$$

The following notation is standard:

$$\mathfrak{n}^+ = \sum_{\alpha \in \triangle^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \sum_{\alpha \in \triangle^-} \mathfrak{g}_\alpha.$$

The subalgebras $\mathfrak{n}^\pm$ are nilpotent.

We put a partial order on $\mathfrak{h}^*$ by saying $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots.

Example 2.3.7 The simplest of the simple Lie algebras is $\mathfrak{sl}(2, \mathbb{C})$, the 2×2 complex matrices of zero trace. Consider the following basis (which we will call the *standard basis*):

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have the relations:

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H.$$

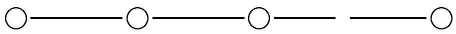
Then $\mathfrak{h} = \mathbb{C} \cdot H$ is a Cartan subalgebra, and there are two roots $\pm\alpha$, with $\alpha(H) = 2$. $\square$

An important aspect of the structure of any semisimple $\mathfrak{g}$ is that it is a sum of copies of $\mathfrak{sl}(2, \mathbb{C})$. Let $\alpha \in \triangle$. Then $\exists\ H_\alpha \in \mathfrak{h}$, $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, s.t. $\mathrm{span}\{H_\alpha, X_\alpha, Y_\alpha\} \cong \mathfrak{sl}(2, \mathbb{C})$ under the map $H_\alpha \mapsto H$, $X_\alpha \mapsto X$, $Y_\alpha \mapsto Y$. For each choice of Ψ this leads to a *Chevalley basis* for $\mathfrak{g}$: $\{X_\alpha, Y_\alpha : \alpha \in \triangle^+\} \cup \{H_{\alpha_i} : \alpha_i \in \Psi\}$.

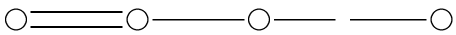
The Killing form of $\mathfrak{g}$ on restriction to $\mathfrak{h}$ remains non-degenerate. So we have an identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ and an inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$. The identification matches $\alpha \in \triangle$ with a multiple of H_α.

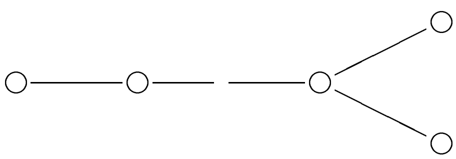
The simple roots and the angles between them classify the simple complex Lie algebras. Thus let $\Psi = \{\alpha_1, \dots, \alpha_n\}$ be a choice of simple roots. The corresponding *Cartan matrix* A is defined by $A_{ij} = 2(\alpha_i, \alpha_j)/|\alpha_i|^2$. Upto permutation of its rows and columns the Cartan matrix characterizes the corresponding simple Lie algebra. The *Dynkin diagram* is a graph in which the i-th node corresponds to the simple root α_i, and the i and j nodes are joined by $A_{ij}A_{ji}$ edges. Moreover, if there are edges between nodes corresponding to roots of different lengths, we add an arrowhead that points to the shorter root. The simple complex Lie algebras are again characterized by their Dynkin diagrams.

The *classical simple Lie algebras* correspond to the following Dynkin diagrams (each with n nodes):

A_n: 

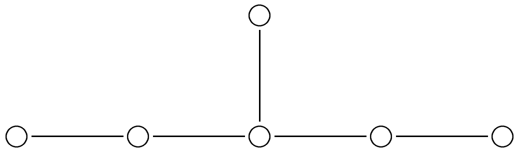
B_n: 

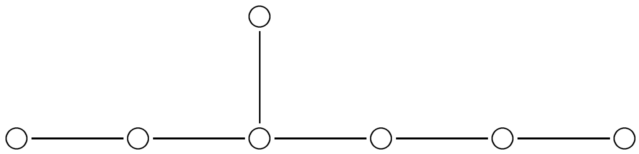
C_n: 

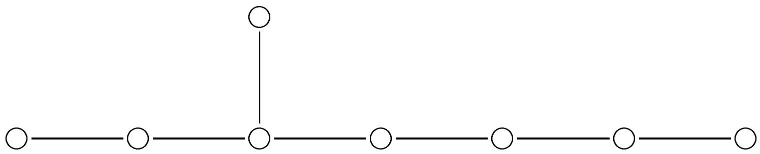
D_n: 

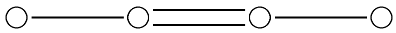
These diagrams match the classical complex Lie algebras we have encountered earlier in the following way: $A_n \sim \mathfrak{sl}(n+1,\mathbb{C})$, $B_n \sim \mathfrak{so}(2n+1,\mathbb{C})$, $C_n \sim \mathfrak{sp}(n,\mathbb{C})$, and $D_n \sim \mathfrak{so}(2n,\mathbb{C})$.

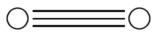
Apart from these infinite families, there are five more simple complex Lie algebras, called the *exceptional Lie algebras*. Their names and Dynkin diagrams are as follows:

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

These diagrams, and in particular the ones for the exceptional algebras, have been included simply because no book on this topic appears complete without them.

Let P_α be the hyperplane in $\mathfrak{h}^*$ orthogonal to $\alpha \in \triangle$. Let σ_α be the reflection across this hyperplane. The group $\mathfrak{W} = \mathfrak{W}(\triangle)$ generated by these reflections is called the *Weyl group*. It preserves $\triangle$, is finite, and acts simply transitively on the set of bases. Further, $\mathfrak{W}$ is in fact generated by $\{\sigma_\alpha | \alpha \in \Psi\}$.

Let $\Gamma(\mathfrak{g})$ be the group of automorphisms of $\triangle$ which preserve the base Ψ. The action of this group on Ψ then must also preserve the edges and arrows of the corresponding Dynkin diagram. If $\mathfrak{g}$ is simple we see from the above list that $\Gamma(\mathfrak{g})$ is non-trivial only for the cases $A_n (n \geq 2)$, $D_n (n \geq 4)$ and E_6.

By $\mathrm{Aut}(\mathfrak{g})$ we denote the group of all Lie algebra automorphisms of $\mathfrak{g}$. $\Gamma(\mathfrak{g})$ can be identified with a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Definition 2.3.8 $\mathrm{Int}(\mathfrak{g})$ is the group of automorphisms of $\mathfrak{g}$ generated by $\{e^{\mathrm{ad}(X)} : X \in \mathfrak{g}\}$. Alternately, it is the analytic subgroup of $GL(\mathfrak{g})$ corresponding to the Lie subalgebra $\mathrm{ad}(\mathfrak{g})$ of $L(\mathfrak{g})$.

We have a semi-direct product:

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Int}(\mathfrak{g}) \cdot \Gamma(\mathfrak{g}).$$

Theorem 2.3.9 *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Cartan subalgebras of $\mathfrak{g}$. Then $\exists\ \tau \in$ $\mathrm{Int}(\mathfrak{g})$ such that $\tau\mathfrak{h}_1 = \mathfrak{h}_2$.* $\square$

Now consider a reductive Lie algebra $\mathfrak{g}$ with $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g}_1$ has a Cartan subalgebra $\mathfrak{h}_1$ and a root decomposition

$$\mathfrak{g}_1 = \mathfrak{h}_1 + \sum_{\alpha \in \triangle(\mathfrak{g}_1, \mathfrak{h}_1)} \mathfrak{g}_\alpha.$$

The decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1$ implies that $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{g}$. The resulting root space decomposition of $\mathfrak{g}$ is simply

$$\mathfrak{g} = \mathfrak{h} + \sum_{\tilde{\alpha} \in \triangle(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{\tilde{\alpha}},$$

with $\triangle = \triangle(\mathfrak{g}, \mathfrak{h}) \cong \triangle(\mathfrak{g}_1, \mathfrak{h}_1)$. The identification of the two root systems is as follows: extend $\alpha \in \triangle(\mathfrak{g}_1, \mathfrak{h}_1) \subset \mathfrak{h}_1^*$ to $\tilde{\alpha} \in \mathfrak{h}^*$ by requiring $\tilde{\alpha}(\mathfrak{z}(\mathfrak{g})) = 0$. Then $\mathfrak{g}_\alpha = \mathfrak{g}_{\tilde{\alpha}}$. The Weyl group $\mathfrak{W}$ acts trivially on $\mathfrak{z}(\mathfrak{g})$.

Consequently, arguments about reductive Lie algebras generally reduce to arguments about semisimple Lie algebras. Instead of stating the appropriate reduction in each situation that comes up, we will assume in the rest of this chapter that $\mathfrak{g}$ is semisimple, and note that all the theorems obtained are also true for the reductive case.

Now consider a $\mathfrak{g}$ module V on which $\mathfrak{h}$ acts semisimply. Then there is a decomposition into common eigenspaces of $\mathfrak{h}$:

$$V = \sum_{\lambda \in \Lambda(V)} V_\lambda,$$

where $\Lambda(V) \subset \mathfrak{h}^*$, $V_\lambda = \{v \in V | Hv = \lambda(H)v\ \forall H \in \mathfrak{h}\}$, and $\lambda \in \Lambda(V)$ if and only if $V_\lambda \neq 0$. The elements of $\Lambda(V)$ are called the *weights of* V. Note that $\mathfrak{g}_\alpha$ maps V_λ into $V_{\lambda+\alpha}$.

A special case of this is the *cyclic module of highest weight* λ. Such a module V has a vector $v \in V_\lambda$ s.t. $\mathfrak{g}_\alpha \cdot v = 0\ \forall \alpha \in \triangle^+$, and $V = \mathcal{U}(\mathfrak{g}) \cdot v$. Here v is a *highest weight vector* and λ is a *highest weight*. For each $\lambda \in \mathfrak{h}^*$, there is a unique irreducible cyclic module of highest weight λ. (See §6.2 for an explicit realization.)

Theorem 2.3.10 *Let $V = V(\lambda)$ be the irreducible cyclic module of highest weight λ. Then $\lambda \neq \mu \in \Lambda(V)$ implies $\lambda > \mu$.* $\square$

An element λ of $\mathfrak{h}^*$ is an *integral weight* if $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\ \forall \alpha \in \triangle$. The set of integral weights is denoted Λ, forms a subgroup of $\mathfrak{h}^*$, and contains $\triangle$. The *root lattice* Λ_r is the subgroup of Λ generated by $\triangle$.

A $\lambda \in \mathfrak{h}^*$ is a *dominant weight* if $\langle \lambda, \alpha \rangle \geq 0\ \forall \alpha \in \Psi$. Λ^+ is the set of dominant integral weights. The $\mathfrak{W}$-orbit of any $\lambda \in \Lambda$ intersects Λ^+ exactly once. If $\lambda \in \Lambda^+$ then there are only finitely many $\mu \in \Lambda^+$ such that $\mu < \lambda$.

Theorem 2.3.11 *Let $V = V(\lambda)$ be the irreducible cyclic module of highest weight λ. It is finite dimensional if and only if λ is a dominant integral weight. Then $\Lambda(V)$ is permuted by $\mathfrak{W}$ and* $\dim(V_\mu) = \dim(V_{\sigma\mu})$ $\forall \sigma \in \mathfrak{W}$. □

2.4 Center of the Enveloping Algebra

Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. Let $G = \mathrm{Int}(\mathfrak{g})$. The action of G on $\mathcal{U}(\mathfrak{g})$ restricts to $\mathcal{Z}$. Recall our recent comment that the succeeding results hold for reductive $\mathfrak{g}$ although we write the proofs only for semisimple $\mathfrak{g}$.

Theorem 2.4.1 $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$.

Proof. (from Humphreys [68]) Consider a generator of G, $e^{\mathrm{ad}(X)}$, with $\mathrm{ad}(X)$ nilpotent, and $Z \in \mathcal{Z}$. Then $\mathrm{ad}(X)Z = [X, Z] = 0$ implies $e^{ad(X)}(Z) = Z$. Since the generators of G fix Z, all of G does.

Now suppose G fixes $X \in \mathcal{U}(\mathfrak{g})$. Note that $\mathfrak{g}$ is generated by its root spaces, so it suffices to show that $[X, X_\alpha] = 0$ for each $X_\alpha \in \mathfrak{g}_\alpha$. Let $n = \mathrm{ad}(X_\alpha)$ and t the largest integer s.t. $n^t \neq 0$. Choose distinct members $a_1, \dots, a_{t+1}$ of $\mathbb{C}$. Then $e^{a_i n} = 1 + a_i n + \cdots + (a_i/t!)n^t$ fixes X. Now

$$\begin{vmatrix} 1 & a_1 & a_1^2/2! & \cdots & a_1^t/t! \\ \vdots & & & & \vdots \\ 1 & a_{t+1} & a_{t+1}^2/2! & \cdots & a_{t+1}^t/t! \end{vmatrix} = (2!3!\dots t!)^{-1} \prod_{i>j}(a_i - a_j) \neq 0.$$

Therefore, the vectors $e^{a_i n}$ have the same span as the vectors $1, \dots, n^t$. In particular, $\exists$ scalars $b_1, \dots, b_{t+1}$ such that $n = \sum_{i=1}^{t+1} b_i(e^{a_i n})$. And then

$$\mathrm{ad}(X_\alpha)X = \sum_{i=1}^{t+1} b_i(e^{a_i n})X = (\sum_{i=1}^{t+1} b_i)X.$$

But $\mathrm{ad}(X_\alpha)$ is nilpotent $\implies \sum_{i=1}^{t+1} b_i = 0 \implies \mathrm{ad}(X_\alpha)X = 0$. □

Let $(X_i)_{i=1}^n$ be a basis of $\mathfrak{g}$ and let $(X^i)_{i=1}^n$ be the dual basis with respect to the Killing form $(\cdot, \cdot)$. Then the Casimir element[12] of $\mathfrak{g}$ is the element of $\mathcal{U}(\mathfrak{g})$ defined by:

$$C = \sum_i X_i X^i.$$

C is independent of the choice of basis. For suppose $(Y_j)_{j=1}^n$ is another basis, with dual basis $(Y^j)_{j=1}^n$. Then we have scalars a_j^i and b_r^j such that $Y_j = \sum_i a_j^i X_i$ and $Y^j = \sum_r b_r^j X^r$. Now

$$\delta_j^k = (Y_j, Y^k) = \sum_{i,r} a_j^i b_r^k (X_i, X^r) = \sum_i a_j^i b_i^k$$

$$\begin{aligned} \implies \sum_j Y_j Y^j &= \sum_j \Big(\sum_i a_j^i X_i\Big)\Big(\sum_r b_r^j X^r\Big) = \sum_{i,j,r} a_j^i b_r^j X_i X^r \\ &= \sum_{i,r} \Big(\sum_j a_j^i b_r^j\Big) X_i X^r = \sum_{i,r} \delta_r^i X_i X^r = \sum_i X_i X^i. \end{aligned}$$

[12]The Casimir element is an incarnation of the Laplacian.

Now suppose $g \in \mathrm{Int}(\mathfrak{g})$. The Killing form is preserved by inner automorphisms, so $(g \cdot X^i)$ is a dual basis to $(g \cdot X_i)$. Therefore C is invariant under g:

$$g \cdot C = \sum_i (g \cdot X_i)(g \cdot X^i) = C,$$

by the previous result on independence of basis. Consequently, $C \in \mathcal{Z}(\mathfrak{g})$.

Example 2.4.2 Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with the standard basis H, X, Y. Calculating the value of the Killing form[13] for different pairs from this basis, we find the only non-zero results are $(H, H) = 8$ and $(X, Y) = 4$. Therefore the dual basis is $H^* = H/8$, $X^* = Y/4$, and $Y^* = X/4$. And the Casimir element is given by

$$C = HH^* + XX^* + YY^* = \frac{H^2}{8} + \frac{XY}{4} + \frac{YX}{4}.$$

We will also find it convenient to work with

$$\Omega = 1 + 8C = 1 + H^2 + 2(XY + YX). \tag{2.3}$$

□

For a finite dimensional vector space V, $\mathcal{P}(V) = \mathcal{S}(V^*)$ is the algebra of *polynomial functions* on V.

Let $G = \mathrm{Int}(\mathfrak{g})$. It acts on $\mathcal{P}(\mathfrak{g})$ by $(\sigma f)(X) = f(\sigma^{-1}X)$. Let $\mathcal{P}(\mathfrak{g})^G$ denote the elements of $\mathcal{P}(\mathfrak{g})$ which are fixed by G. Also, $\mathfrak{W}$ acts on $\mathfrak{h}^*$ and hence on $\mathcal{P}(\mathfrak{h})$. Let $\mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$ be the polynomial functions fixed by $\mathfrak{W}$.

Any polynomial function on $\mathfrak{g}$, when restricted to $\mathfrak{h}$, is a polynomial function on $\mathfrak{h}$. So we have the restriction map $\theta : \mathcal{P}(\mathfrak{g}) \to \mathcal{P}(\mathfrak{h})$.

Theorem 2.4.3 (Chevalley) *The map θ is an algebra isomorphism of $\mathcal{P}(\mathfrak{g})^G$ onto $\mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$.*

Proof. Suppose $f \in \mathcal{P}(\mathfrak{g})^G$. Then it is fixed by the inner automorphism $\tau_\alpha = \exp \mathrm{ad}(x_\alpha) \exp \mathrm{ad}(-y_\alpha) \exp \mathrm{ad}(x_\alpha)$, and so $\theta(f) = f|_{\mathfrak{h}}$ is fixed by $\tau_\alpha|_{\mathfrak{h}} = \sigma_\alpha$. So $\theta(f)$ is fixed by $\mathfrak{W}$. Here x_α, y_α, are members of a Chevalley basis as described in Section 2.3.

We skip the proof of surjectivity. It consists of explicit constructions of elements of $\mathcal{P}(\mathfrak{g})^G$ which map onto elements $sym(\lambda^k)$, $\lambda \in \Lambda^+$, and these span $\mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$. [14]

Now suppose $f \in \mathcal{P}(\mathfrak{g})^G$ such that $f|_{\mathfrak{h}} = 0$. Then $f = 0$ on all the images of $\mathfrak{h}$ under G, i.e. on all the Cartan subalgebras. Therefore $f = 0$ on the regular

[13]The calculations are easier if we first observe that the Killing form (A,B) must be a constant times the form $\mathrm{Tr}\,(AB)$ since both are non-degenerate and invariant. The constant is easily calculated to be 4 by looking at $\mathrm{Tr}\,(H^2)$.

[14]For the specifics see [68].

semisimple elements of $\mathfrak{g}$ (as each of them lies in a Cartan subalgebra). But these are dense in $\mathfrak{g}$, and hence $f = 0$ on $\mathfrak{g}$. $\square$

Recall that $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$, where $G = \mathrm{Int}(\mathfrak{g})$. Also, the Poincaré-Birkhoff-Witt theorem implies $\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$ (as linear spaces). This is actually an isomorphism of G modules, and so $\mathcal{U}(\mathfrak{g})^G \cong \mathcal{S}(\mathfrak{g})^G$. The Killing form identifies $\mathfrak{g}$ and $\mathfrak{h}$ with $\mathfrak{g}^*$ and $\mathfrak{h}^*$, hence $\mathcal{S}(\mathfrak{g})^G$ with $\mathcal{P}(\mathfrak{g})^G$, and $\mathcal{U}(\mathfrak{h})^{\mathfrak{W}} \cong \mathcal{S}(\mathfrak{h})^{\mathfrak{W}}$ with $\mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$. So Chevalley's theorem leads to the following sequence: $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G \longrightarrow \mathcal{S}(\mathfrak{g})^G \longrightarrow \mathcal{P}(\mathfrak{g})^G \xrightarrow{\theta} \mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$, where all the maps are linear isomorphisms, and the last two are also algebra isomorphisms.

Let us look at the case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. As usual, consider the standard basis H, X, Y, and the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\cdot H$. Now we can use the reduction theorem to describe $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$.

Corollary 2.4.4 $\mathcal{Z}(\mathfrak{sl}(2,\mathbb{C})) = \mathbb{C}[\Omega]$, *where* $\Omega = 1 + H^2 + 2(XY + YX)$.

Proof. Ω corresponds to the polynomial ω on $\mathfrak{g}$ given by the same formula. And then $\theta(\omega) = 1 + H^2$. The Weyl group $\mathfrak{W}$ here consists of two elements and the non-trivial one maps $H \mapsto -H$. Therefore $\mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$ is generated by H^2 — or $1 + H^2$. Since the last two maps in the sequence were algebra isomorphisms, this means $\mathcal{S}(\mathfrak{g})^G$ is generated by ω. Now consider $Z \in \mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$. This is the image of some $p(\omega) \in \mathbb{C}[\omega] = \mathcal{S}(\mathfrak{g})^G$. Let $p(\omega) = a_o\omega^k + a_1\omega^{k-1} + \cdots a_k$. Then $Z = a_o\Omega^k + Z_1$ where Z_1 is of lower degree than Z. Repeat the process with Z_1. By induction we see that Z is a polynomial in Ω. $\square$

Now we describe a different characterization of the center $\mathcal{Z}$. In a sense that the discussion below will make clearer, this is a perturbation of the description given by the Chevalley theorem. The defect we seek to correct is that use of that theorem involves the identification $\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$, which is not an algebra homomorphism.

Consider the projection from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{h})$ given by the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n})$ and let p be its restriction to $\mathcal{Z}$. Then p can be shown to be an algebra homomorphism. Further, the image of p lies in $\mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}$.

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and choose the positive root to be given by $\alpha(H) = 2$. Look at Ω again. We can express it also as $\Omega = (H+1)^2 + 4YX = (H-1)^2 + 4XY$. Then $p(\Omega) = (H+1)^2$. Now Ω is an invariant of the algebra[15], but this image $p(\Omega)$ is not. Of course, it must depend on the choice of $\mathfrak{h}$, but we see it also depends on the choice of the positive root: the other choice would give the image $(H-1)^2$. So we are led to a correcting factor. Define $u(H) = H - \rho(H)\cdot 1$, where ρ is half the sum of the positive roots, and extend to a homomorphism $u : \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$. Let $\gamma = u \circ p$. Then $\gamma(\Omega) = H^2$ and this is independent of the choice of positive root. Another way of saying this is that it is invariant under the Weyl group.

[15] It equals 1 + a multiple of the Casimir element.

Let's return to the general set-up. Now $u : \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h})$ is defined by extending from the expression $u(H) = H - \rho(H) \cdot 1$, $\forall H \in \mathfrak{h}$. Let $\gamma = u \circ p$.

Theorem 2.4.5 (Harish-Chandra) *The map γ is injective on $\mathcal{Z}$ and in fact is an algebra isomorphism of $\mathcal{Z}$ onto $\mathcal{U}(\mathfrak{h})^{\mathfrak{W}}$.*

Proof. We write only a general description of the proof. The precise details can be found in many references, for instance [145]. First, let's see why γ maps $\mathcal{Z}$ into $\mathcal{U}(\mathfrak{h})^{\mathfrak{W}}$. It is enough to show that the image is invariant under the simple reflections σ_α, $\alpha \in \Psi$. Acting on the image by σ_α has the same result as starting with the base $\sigma_\alpha\Psi$ instead of Ψ. The resulting change in p is now exactly corrected by the change in ρ.

The identifications described in the previous section lead to the following diagram. If it commutes, we are done.

$$\begin{tikzcd}
\mathcal{S}(\mathfrak{g})^G \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{Z} \arrow[r, "\gamma"] & \mathcal{U}(\mathfrak{h})^{\mathfrak{W}} \arrow[d, leftrightarrow] \\
\mathcal{P}(\mathfrak{g})^G \arrow[rr, "\theta"] & & \mathcal{P}(\mathfrak{h})^{\mathfrak{W}}
\end{tikzcd}$$

Consider the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let us start with $\Omega \in \mathcal{Z}$, which maps to $H^2 \in \mathcal{U}(\mathfrak{h})^{\mathfrak{W}}$, and then to $H^{*2} \in \mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$. In the other direction, Ω corresponds to $H^2 + 1 + 2XY + 2YX \in \mathcal{S}(\mathfrak{g})^G$ and hence to $H^{*2} + 1 + 2X^*Y^* + 2Y^*X^* \in \mathcal{P}(\mathfrak{g})^G$. Under θ that maps to $H^{*2} + 1 \in \mathcal{P}(\mathfrak{h})^{\mathfrak{W}}$.

So the diagram is not commutative. On the other hand, we see that the failure is measured by a term of lower degree. This holds in general, by an argument analogous to the one used in Corollary 2.4.4. Therefore we can push the argument for surjectivity through by induction on degree. $\square$

The map γ is known as the *Harish-Chandra isomorphism.* He published it in [45], a famous work where he also constructed the simple Lie algebras in a uniform manner (rather than case by case) as well as their highest weight representations.

Infinitesimal Character

Let (π, V) be a representation of $\mathcal{U}(\mathfrak{g})$, for a complex reductive $\mathfrak{g}$. We say that it "has an infinitesimal character" if the center $\mathcal{Z}$ of $\mathcal{U}(\mathfrak{g})$ acts by scalars. Then we get an algebra homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$ defined by

$$\pi(Z)(v) = \chi(Z)v,$$

and this is called the *infinitesimal character* of V.

The Harish-Chandra isomorphism helps us understand the possibilities for χ. Let $\lambda \in \mathfrak{h}^*$. It extends to a homomorphism from $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. Define $\chi_\lambda(Z) = \lambda(\gamma(Z))$. This is a homomorphism from $\mathcal{Z}$ into $\mathbb{C}$. Let $w \in \mathfrak{W}$. Then $w \circ \gamma = \gamma \implies \chi_{w\lambda} = \chi_\lambda$.

Theorem 2.4.6 *The following are consequences of Theorem 2.4.5:*

1. Let $\lambda, \mu \in \mathfrak{h}^*$. Then $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same $\mathfrak{W}$-orbit.

2. Let $\chi : \mathcal{Z} \to \mathbb{C}$ be a homomorphism. Then $\exists\, \lambda \in \mathfrak{h}^*$ such that $\chi = \chi_\lambda$. □

Example 2.4.7 Suppose $V(\lambda)$ is the irreducible cyclic $\mathfrak{g}$ module with highest weight λ. Let $v \in V_\lambda$ be a highest weight vector. Let $Z \in \mathcal{Z}$. Then $Z = p(Z) + X$, where $X \in \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Note also that $\lambda \circ p = (\lambda + \rho) \circ \gamma$. This yields

$$Z \cdot v = (p(Z) + X) \cdot v = p(Z) \cdot v = \lambda(p(Z))v = (\lambda + \rho)(\gamma(Z))v,$$

so that the infinitesimal character of this representation is $\chi_{\lambda+\rho}$.[16] □

Example 2.4.8 Let G be a reductive Lie group with maximal compact subgroup K, and let V be an admissible $(\mathfrak{g}, K)$ module. Then each K-isotype $V(\xi)$ is also $\mathcal{Z}$-stable and $\mathcal{Z}|_{V(\xi)}$ can be put in upper triangular form. For any homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$, define

$$V_\chi = \{v \in V : \ (z - \chi(z))^d v = 0, \ \forall z \in \mathcal{Z}, \ d = d(v)\}.$$

Then V is an algebraic direct sum of the V_χ as $(\mathfrak{g}, K)$ modules. If V is irreducible, then we get $V = V_\chi$ for some χ. Now let $W_\chi = \{v \in V : (z - \chi(z))v = 0, \ \forall z \in \mathcal{Z}\}$. This is a non-zero $(\mathfrak{g}, K)$ module, hence $V = W_\chi$. Thus χ is the infinitesimal character of V. □

2.5 Compact Lie Groups

Our discussion of compact groups so far has not exploited their possible differential structure. Let us fix a compact connected real Lie group K with Lie algebra $\mathfrak{k}_o$. The first result is that such a group can be realized as a matrix group.

Theorem 2.5.1 *Every compact Lie group K has a faithful finite dimensional representation.*

Proof. The Peter-Weyl theorem implies that the irreducible unitary representations separate points of K, and these representations are finite dimensional. So for each $x \neq 1$ in K there is a finite dimensional representation π_x of K such that $\pi_x(x) \neq I$.

Let ψ_1 be any finite dimensional representation of K. If it is not faithful, pick an $x_1 \neq 1$ in $\ker \psi_1$ and define $\psi_2 = \psi_1 \oplus \pi_{x_1}$. Then $\ker \psi_2$ is a proper closed subgroup of $\ker \psi_1$. Proceeding in this way we get a sequence of finite dimensional representations ψ_n such that $\ker \psi_n$ forms a strictly decreasing sequence of closed subgroups of K. At each step either the number of components or the dimension must strictly decrease. Therefore the sequence must terminate at some ψ_N such that $\ker \psi_N$ is trivial. □

[16] For this reason, some authors prefer to use χ_λ for the infinitesimal character corresponding to $\lambda + \rho$.

Since $\mathfrak{k}_o$ (and hence the complexification $\mathfrak{k}$) is reductive, our previous definitions of Cartan subalgebras and root systems apply and yield the irreducible representations of $\mathfrak{k}$ by way of the highest weight theory. The question now is—which of these lift to representations of K?

The first task is to identify the subgroups of K which correspond to the Cartan subalgebras of $\mathfrak{k}_o$.

Definition 2.5.2 A *torus* is a finite product of circle groups: $S^1 \times \cdots \times S^1$. Every connected compact abelian Lie group is isomorphic to a torus.

Theorem 2.5.3 *The maximal tori of the compact Lie group K are the analytic subgroups corresponding to the maximal abelian subalgebras of its Lie algebra $\mathfrak{k}_o$.*

Proof. Let T be a maximal torus of K, and let $\mathfrak{t}_o$ be its Lie algebra. Let $\mathfrak{h}_o$ be a strictly larger abelian subalgebra. Then the analytic subgroup H corresponding to $\mathfrak{h}_o$ is abelian, and its closure $\bar{H}$ is a torus strictly containing T.

Now suppose $\mathfrak{t}_o$ is a maximal abelian subalgebra of $\mathfrak{k}_o$. Let T be its analytic subgroup. If T is not closed, its closure $\bar{T}$ has a strictly larger (abelain) Lie algebra than $\mathfrak{t}_o$. Hence T is closed, and so it is compact. The same reasoning shows it is maximal. □

The conjugacy of the Cartan subalgebras and their containing the regular elements has the following analogue for compact Lie groups:[17]

Theorem 2.5.4 *Let T be a maximal torus in K. Then*

1. *The conjugates of T cover K.*
2. *T is a maximal abelian subgroup of K.*
3. *Every maximal torus in K is conjugate to T.*
4. *T is equal to its centralizer.*[18] □

Due to the above features, maximal tori are called *Cartan subgroups* of K. Now fix a maximal torus T of K, and let $\mathfrak{t}_o$ be its Lie algebra. Then the complexification $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Let $\triangle = \triangle(\mathfrak{k}, \mathfrak{t}) \subset \mathfrak{t}^*$ be the corresponding root system.

Since K is compact we can give $\mathfrak{k}$ an $\mathrm{Ad}(K)$ invariant inner product. Relative to this inner product $\mathrm{Ad}(T)$ is a family of commuting unitary transformations, hence $\mathfrak{k}_o$ decomposes into simultaneous eigenspaces of $\mathrm{Ad}(T)$. This decomposition must match the root space decomposition under $\mathrm{ad}(\mathfrak{t})$:

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \triangle} \mathfrak{k}_\alpha.$$

[17]For the structure of compact Lie groups see Hochschild [64, Ch XIII] or Knapp [78, Ch IV].

[18]This is a corollary of the following: The centralizer of *any* torus in K is connected.

The action of T on $\mathfrak{k}_\alpha$ is by a character ζ_α and $\alpha|_{\mathfrak{t}_o}$ must be the differential of this character.

Lemma 2.5.5 *If $\alpha \in \triangle(\mathfrak{k}, \mathfrak{t})$, then α takes purely imaginary values on $\mathfrak{t}_o$.*

Proof. If $H \in \mathfrak{t}_o$ we have $e^{\alpha(H)} = \zeta_\alpha(e^H) \in S^1 \subset \mathbb{C}$. □

If a representation of $\mathfrak{k}$ arises from a representation of K then the weights must be differentials of characters of T.

Definition 2.5.6 A weight $\lambda \in \mathfrak{t}^*$ is *analytically integral* if $\lambda|_{\mathfrak{t}_o}$ is the differential of a character ζ_λ of T. It follows that:

(1) $\zeta_\lambda(e^H) = e^{\lambda(H)}$ for every $H \in \mathfrak{t}_o$.

(2) $H \in \mathfrak{t}_o$ and $e^H = 1 \implies \lambda(H) \in 2\pi i\mathbb{Z}$.

In fact (2) implies that λ is analytically integral. Moreover, if λ is analytically integral then it is an integral weight in the sense of §2.3.

Theorem 2.5.7 *Fix a choice of positive roots $\triangle^+$ of $\triangle(\mathfrak{k}, \mathfrak{t})$. Let $\lambda \in \mathfrak{t}^*$ be a dominant integral weight relative to $\triangle^+$. Then the irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight λ lifts to a representation of K if and only if λ is analytically integral.* □

The next task is to have a systematic construction of the irreducible unitary representations of K. It is natural to consider some kind of induction starting from characters of T. In fact we will describe a holomorphic analogue of Mackey induction. We start by giving K/T a complex structure via Weyl's "unitary trick".

Definition 2.5.8 A *complexification* of a real Lie group G is a complex Lie group $G_\mathbb{C}$ which contains G as a subgroup and whose Lie algebra is the complexification of the Lie algebra of G.

Theorem 2.5.1, and the fact that every finite dimensional representation of K can be made unitary, together imply that K can always be realized as a closed subgroup of some unitary group $U(n)$. Then the analytic subgroup $K_\mathbb{C}$ of $GL(n, \mathbb{C})$ which corresponds to $\mathfrak{k} \subset \mathfrak{gl}(n, \mathbb{C})$ is a complexification of K. For the rest of this section, we write $G = K_\mathbb{C}$, so that $\mathfrak{g}_o$ is $\mathfrak{k}$ viewed as a real Lie algebra. The complexification G is unique up to isomorphism and is a linear reductive group with K as a maximal compact subgroup.[19]

Theorem 2.5.9 *If (π, V) is a continuous finite dimensional representation of K, then π extends to a holomorphic representation of G in V.* □

Fix a positive system of roots $\triangle^+ \subset \triangle = \triangle(\mathfrak{k}, \mathfrak{t})$, and let $\mathfrak{b}^\pm$ be the resulting Borel subalgebras. Let $B^\pm$ denote the corresponding subgroups of G. Then $B^\pm \cap K = T$ and $K/T \cong G/B^- \cong G/B^+$. Thus K/T has a complex structure.

[19] The table following Definition 2.1.18 contains some examples.

Theorem 2.5.10 *The complex manifold K/T is simply connected and Kähler. All simply connected compact Kähler homogeneous spaces are of this type.* □

Lemma 2.5.11 *Let $\lambda \in \mathfrak{t}^*$ be analytically integral. Then ζ_λ extends continuously to a holomorphic homomorphism from B^- to $\mathbb{C}^\times$.*

Proof. Let $\mathfrak{n}^- = \sum_{\alpha \in \triangle^-} \mathfrak{k}_\alpha$, so that $\mathfrak{b}^- = \mathfrak{t} \oplus \mathfrak{n}^-$. Let A be the connected real Lie subgroup of G corresponding to $i\mathfrak{t}_o$, and N^- the subgroup corresponding to $\mathfrak{n}^-$. Then $B^- = TAN^-$. (While $G = KAN^-$ is an Iwasawa decomposition of G.) Extend ζ_λ to B^- as follows:

$$\zeta_\lambda(x) = \begin{cases} \exp\{\lambda(\log(x))\} & x \in A \\ 1 & x \in N^- \end{cases}$$

□

If we denote the corresponding representation of B^- in $\mathbb{C}$ by $\mathbb{C}_\lambda$, then $L_\lambda = G \times_{B^-} \mathbb{C}_\lambda$ defines a homogeneous holomorphic line bundle on $G/B^- \cong K/T$. ($G \times_{B^-} \mathbb{C}_\lambda$ is $G \times \mathbb{C}_\lambda$ modulo the equivalence relation $(gb, z) \sim (g, \zeta_\lambda(b)^{-1}z)$ for any $g \in G$ and $b \in B^-$.) Let $\Gamma(\lambda)$ be the space of global holomorphic sections of L_λ, with G acting by left translations. An alternate realization of $\Gamma(\lambda)$ is as the following function space:

$$\Gamma(\lambda) = \{f : G \xrightarrow{\text{holom}} \mathbb{C} : f(gb) = \zeta_\lambda(b)^{-1} f(g) \ \forall g \in G,\ b \in B^-\},$$

with G again acting by left translations.

Theorem 2.5.12 (Borel-Weil) *Let $\lambda \in \mathfrak{t}^*$ be analytically integral. Then $\Gamma(\lambda)$ is non-zero if and only if λ is a dominant weight of T. If λ is a dominant weight then $\Gamma(\lambda)$ is the irreducible finite dimensional representation of G with highest weight λ.* □

The Borel-Weil theorem constructs all the irreducible finite dimensional representations of K and G. To simplify our results we now assume K (and hence G) is simply connected. Then integrable weights are automatically analytically integrable. For future reference, we record the following definitions:

Definition 2.5.13 Let $\mathfrak{g}$ be a complex reductive Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and corresponding root system $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$. A weight $\lambda \in \mathfrak{h}^*$ is *regular* if $(\lambda, \alpha) \neq 0 \ \forall \alpha \in \triangle$. Else, it is *non-regular.*

Definition 2.5.14 In the same situation as above, and with ρ denoting the half-sum of a choice of positive roots, $\lambda \in \mathfrak{h}^*$ is *singular* if $\lambda + \rho$ is non-regular. It is *non-singular* otherwise.[20]

Definition 2.5.15 Let $w \in \mathfrak{W}(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra of a complex reductive Lie algebra $\mathfrak{g}$, and Ψ a choice of simple roots. The *length* of w, denoted $\ell(w)$, is the least integer n such that $w = s_{\alpha_1} \cdots s_{\alpha_n}$ where the $\alpha_i \in \Psi$ and the s_{α_i} are the corresponding reflections.

[20] This distinction between *regular* and *non-singular* appears to be common but not universal.

The Borel-Weil theorem has the following generalization due to Bott [13]. (See also Kostant [85])

Theorem 2.5.16 (Bott-Borel-Weil) *Let G be a simply connected complex semisimple group. With notation as above, let $\lambda \in \mathfrak{h}^*$ be an integral weight, and let $\mathcal{O}(\lambda)$ denote the sheaf of germs of holomorphic sections of L_λ. Then*

1. *If λ is singular, then $H^r(G/B^-, \mathcal{O}(\lambda)) = 0 \ \forall r$.*
2. *If λ is non-singular, let $w \in \mathfrak{W}(\mathfrak{g}, \mathfrak{h})$ be the unique member such that $\mu = w(\lambda + \rho) - \rho$ is dominant. Then*

$$H^r(G/B^-, \mathcal{O}(\lambda)) = \begin{cases} F(\mu) & r = \ell(w) \\ 0 & \text{else} \end{cases},$$

where $F(\mu)$ is the finite dimensional irreducible representation of G with highest weight μ. □

Baston and Eastwood [9, Ch 5] has a lucid exposition of this result. Proofs of the Borel-Weil and Bott-Borel-Weil theorems can also be found in Knapp [76] and Knapp and Vogan [81], respectively. In fact, the Bott-Borel-Weil theorem exists in a more general setting, where it addresses induction from "parabolic" subgroups—more general objects than the Borel subgroups used here. Although the Bott-Borel-Weil theorem does not construct any new representations (compared to Borel-Weil), it does give new realizations of those representations and it is these that generalize best to the non-compact case.

To complete the description of $\widehat{K}$ we need the characters and dimensions of the irreducible representations. First define

$$\delta = \prod_{\alpha \in \triangle^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\rho} \prod_{\alpha \in \triangle^+} (1 - e^{-\alpha}).$$

Since the roots and ρ are integral they are analytically integral and so δ, originally defined on $\mathfrak{t}$, is in fact well-defined on T.

Theorem 2.5.17 (Weyl Character Formula) *Let K be a simply connected compact group, T a maximal torus, $\mathfrak{W}_c = \mathfrak{W}(\mathfrak{k}, \mathfrak{t})$ and χ_λ the character of the irreducible representation of K with highest weight λ. Then*

$$\chi_\lambda\big|_T(t) = \frac{\sum_{w \in \mathfrak{W}_c} (\det w) e^{w(\lambda+\rho)}}{\delta(t)}.$$

□

Corollary 2.5.18 (Weyl Denominator Formula) *Evaluating the above for the trivial representation we get*

$$\delta(t) = \sum_{w \in \mathfrak{W}_c} (\det w) e^{w(\rho)}.$$

□

Theorem 2.5.19 (Weyl Dimension Formula) *With the same notation as above, let d_λ be the dimension of the irreducible representation of K with highest weight λ. Then*

$$d_\lambda = \frac{\prod_{\alpha\in\triangle^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\triangle^+}(\rho,\alpha)}.$$

$\square$