Chapter 7

Overview

In this chapter we survey the landscape. We have established the main construction of representations–parabolic induction–and done the standard examples, namely $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$. Now we describe some of the general appearance and state of representation theory for real reductive groups. A wonderful collection of expository articles on these topics is to be found in the proceedings of the 1996 instructional conference in Edinburgh [3].

We free ourselves from presenting the material in an order reflecting the history of the material or any sequence of logical deductions.

7.1 The Subrepresentation Theorem

In this section we mainly follow the exposition in Collingwood [16], which is a review essay on Wallach [145]. Let us fix a standard minimal parabolic $P = MN$, with $M = MA$ called a *Levi subgroup* of G. A fundamental result of Harish-Chandra, the *Subquotient Theorem* [47], asserts that any irreducible representation π of G is infinitesimally equivalent to a subquotient (ie, a quotient of a subspace) of a principal series representation $U(P,\sigma,\lambda)$. This was refined by Casselman to the *Subrepresentation Theorem*:

Theorem 7.1.1 *Every irreducible $(\mathfrak{g},K)$-module is equivalent to a subrepresentation of some $U(P,\sigma,\lambda)$.* □

Let us recall that induction takes us from admissible representations of reductive subroups M to admissible representations of G. To get a better picture of the situation, we also need the reverse process of reduction: going from admissible representations of G to admissible representations of M. Consider the functor that takes an admissible representation $(\pi,\mathcal{H})$ of G with underlying Harish-Chandra module $\mathcal{H}_K$, to the finite dimensional space $H_0(\mathfrak{n},\pi) = \mathcal{H}_K/(\mathfrak{n}\cdot\mathcal{H}_K)$ of $\mathfrak{n}$-coinvariants in π. Let $U(P,\sigma,\lambda)$ have representation space $\mathcal{V}$, induced from $\sigma\otimes e^{\lambda}\otimes 1$ acting on $\mathcal{V}^{\sigma}$. The two processes are related by *Frobenius reciprocity*:

$$\mathrm{Hom}_{\mathfrak{g},K}(\mathcal{H}_K,\mathcal{V}_K) = \mathrm{Hom}_{\mathfrak{m},M}(H_0(\mathfrak{n},\pi),\mathcal{V}^{\sigma}_M).$$

The key to the embedding result is the following:

Theorem 7.1.2 *If π is irreducible then $H_0(\mathfrak{n}, \pi) \neq 0$.* □

Viewing $H_0(\mathfrak{n}, \pi)$ as a finite-dimensional representation of the abelian group A, there must be A-invariant spaces

$$0 = S_0 \subset S_1 \subset \cdots \subset S_t = H_0(\mathfrak{n}, \pi)$$

such that each S_j/S_{j-1} is a one-dimensional character $e^{\nu(\log a)}$, $a \in A$, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. We define the space of *homology exponents of* π:

$$\mathcal{A}_{hom}(\pi) = \{\nu \ : \ e^{\nu(\log a)} \text{ arises as a subquotient } S_j/S_{j-1} \text{ of } H_0(\mathfrak{n}, \pi)\}$$

Casselman's approach to proving $\mathcal{A}_{hom}(\pi) \neq 0$ involves analysis of the matrix coefficients of $\mathcal{H}_K$. Write such a matrix coefficient as $f_{v,w}(g) = \langle w, \pi(g)v \rangle$, $g \in G$, $v \in \mathcal{H}_K$, and $w \in \hat{\mathcal{H}}_K$. [1] It turns out to have an expansion in exponentials of form $e^{\nu(\log a)}$; so one can define

$$\mathcal{A}_{asy}(\pi) = \{\nu \ : \ e^{\nu(\log a)} \text{ arises in the expansion of some } f_{v,w} \text{ for } \pi\}$$

Casselman found non-zero elements of $\mathcal{A}_{asy}(\pi)$ that were also in $\mathcal{A}_{hom}(\pi)$. These *leading exponents* provide the λ we seek. If ν is such a leading exponent, let $\lambda = \nu + \rho$. This leading exponent also provides a non-zero G map from H into the space of the induced representation

$$\mathrm{Ind}^G_{AN}(e^\lambda \otimes 1) = \mathrm{Ind}^G_{MAN}[(\mathrm{Ind}^M_1(1)) \otimes e^\lambda \otimes 1]$$

Now $\mathrm{Ind}^M_1(1) = L^2(M)$ decomposes discretely by the Peter-Weyl Theorem. So we get a G map into $U(P, \sigma, \lambda)$ for some σ. The map is injective because $\mathcal{H}$ is irreducible.

7.2 Global Characters and Discrete Series

Let $(\pi, \mathcal{H})$ be a Hilbert space representation of G, and consider $f \in C_c^\infty(G)$. Let $\pi(f)$ be the corresponding bounded operator on $\mathcal{H}$. We have the following general version of Theorem 3.9.1.

Theorem 7.2.1 *If $(\pi, \mathcal{H})$ is finitely generated admissible and $f \in C_c^\infty(G)$ then $\pi(f)$ is an operator of trace class and $\Theta_\pi : f \mapsto \mathrm{Tr}\,\pi(f)$ is a distribution.*[2] □

In the above situation we say the representation π has *global character* Θ_π.

Corollary 7.2.2 *Real reductive groups are CCR.*

Proof. The proof of Theorem 3.9.2 goes through. □

If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are admissible finitely generated and their underlying $(\mathfrak{g}, K)$ modules are isomorphic, then $\Theta_\pi = \Theta_{\pi'}$. So if V is the underlying $(\mathfrak{g}, K)$ module, we can write Θ_V for Θ_π.

[1] $\hat{\mathcal{H}}_K$ is the K-finite dual of $\mathcal{H}_K$.

[2] Lemma 8.1.1 of [145].

Theorem 7.2.3 *Let $\{\pi_i\}_{i=1}^3$ be representations of G with global characters Θ_i and underlying Harish-Chandra modules V_i such that $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of $(\mathfrak{g}, K)$ modules. Then $\Theta_2 = \Theta_1 + \Theta_3$.* □

This shows that the global characters cannot distinguish between quotients and subrepresentations. On the other hand, the following consequence[3] establishes that the global characters exactly determine infinitesimal equivalence among irreducible representations:

Corollary 7.2.4 *Let $\{\pi_i\}_1^n$ be infinitesimally inequivalent irreducible admissible representations with global characters $\{\Theta_i\}_1^n$. Then the Θ_i are linearly independent.* □

Consider the infinitesimal right translation action of $\mathcal{U}(\mathfrak{g})$ on $C_c^\infty(G)$ given by

$$(X \cdot f)(x) = \frac{d}{dt}\Big|_{t=0} f(xe^{tX}) \quad \text{for } X \in \mathfrak{g}_0.$$

Then $\mathcal{U}(\mathfrak{g})$ acts on the distributions on $C_c^\infty(G)$ as follows: $(u \cdot \Theta)(f) = \Theta(u^t \cdot f)$, $u \in \mathcal{U}(g)$, $f \in C_c^\infty(G)$.

Lemma 7.2.5 *If π has global character Θ_V and infinitesimal character χ then $z \cdot \Theta_V = \chi(z)\Theta_V$ $\forall z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$.*

Proof. Let $X \in \mathfrak{g}_0$. Then a quick calculation shows that $\pi(X \cdot f) = \pi(f)\pi(-X)$. It follows that for $u \in \mathcal{U}(\mathfrak{g})$ we have $\pi(u \cdot f) = \pi(f)\pi(u^t)$. Thus we get:

$$z \cdot \Theta_V(f) = \Theta_V(z^t \cdot f) = \operatorname{Tr} \pi(z^t \cdot f) = \operatorname{Tr}(\pi(f)\chi(z)) = \chi(z)\Theta_V(f).$$

□

Due to this property Θ_V is called an *invariant eigendistribution* [4] *with infinitesimal character* χ.

Definition 7.2.6 A measurable function f on G is *locally integrable* if for every $g \in C_c(G)$ we have $fg \in L^1(G)$.

Definition 7.2.7 Let l be the rank of $\mathfrak{g}$. Then $x \in G$ is *regular* if the multiplicity of 1 as a root of $\det(\mathrm{Ad}(x))$ is exactly l.

The set G' of regular points in G is open and dense, with its complement being of lower dimension. Practical use of global characters is made possible by Harish-Chandra's *Regularity Theorem* [51]:

Theorem 7.2.8 *Any invariant eigendistribution on G is given by a locally integrable function whose restriction to the regular set G' of G is real analytic.* □

For a Cartan subgroup H of G, let $H' = H \cap G'$. Then $G' = \cup_H H'$, where H varies over the Cartan subgroups of G. The analysis of an invariant eigendistribution Θ can be shown to reduce to analysis of Θ over each H' and then to knowing Θ over one representative from each conjugacy class

[3] Lemma 8.1.4 of [145].

[4] Or *central* eigendistribution.

of Cartan subgroups. This kind of reduction is achieved by the tools called *orbital integrals.*

Since matrix coefficients do not have compact support, we have to expand our set of test functions. The Sobolev space defined below suffices to determine the discrete series representations through their characters. Later we will give Harish-Chandra's definition of Schwartz space for semisimple Lie groups, and that will be enough to describe the reduced dual. Harmonic analysis on these spaces, and for L^p $(p \neq 2)$ were first carried out by Ehrenpreis and Mautner in the context of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ [25, 26, 27]. An exposition of the details of such matters is in the survey article by Varadarajan [130].

Let $\mathcal{U}_n(\mathfrak{g})$ be the set of elements of $\mathcal{U}(\mathfrak{g})$ of order less than or equal to n. Consider the family $\{p_u\}_{u\in\mathcal{U}(\mathfrak{g})}$, of seminorms on $C_c^\infty(G)$ defined by $p_u(f) = ||u\cdot f||_2$, where u has the right action on $C_c^\infty(G)$. The completion with respect to this family of seminorms is called the nth Sobolev space on G and can be expressed as:

$$S_n(G) = \{f \in L^2(G) : \ u \cdot f \in L^2(G) \ \forall u \in \mathcal{U}_n(\mathfrak{g})\}.$$

Theorem 7.2.9 *If π is a discrete series representation of G then Θ_π extends to a bounded linear functional on $S_n(G)$ for some n.* □

Recall the following formal expression (after fixing a Cartan subgroup H and a choice of positive roots $\triangle^+$):

$$\delta = \prod_{\alpha\in\triangle^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha\in\triangle^+} (1 - e^{-\alpha}).$$

Since 2ρ is a weight, $|\delta|$ is a well-defined function on H.

Definition 7.2.10 An invariant eigendistribution Θ *decays at* ∞ if for every Cartan subgroup H, $\Theta\big|_H|\delta|$ goes to 0 outside compact subsets of H.

Theorem 7.2.11 *Let Θ be an invariant eigendistribution on G. If Θ extends continuously to some $S_n(G)$ then it decays at ∞.* □

Corollary 7.2.12 *If G has a discrete series representation then it has a compact Cartan subgroup.* □

For the corollary one considers the restriction of Θ to the maximal compact subgroup and shows that the growth requirements can only be true if it is compact.

Theorem 7.2.13 (Harish-Chandra [52]) *Let T be a compact Cartan subgroup of G, $\mathfrak{W}_c = \mathfrak{W}(\mathfrak{k},\mathfrak{t})$, λ a weight of the pair $(\mathfrak{g},\mathfrak{t})$ which is non-singular with respect to some choice of $\triangle^+$. Then there exists a unique invariant eigendistribution $\Theta_{\lambda+\rho}$ such that*

1. *$\Theta_{\lambda+\rho}$ decays at ∞.*
2. $$\Theta_{\lambda+\rho}\big|_T = (-1)^q \frac{\sum_{w\in\mathfrak{W}_c} \det(w) e^{w(\lambda+\rho)}}{\prod_{\alpha\in\triangle,(\lambda+\rho,\alpha)>0}(e^{\alpha/2} - e^{-\alpha/2})},$$

where $q = \dim(G/K)$. Further, the $\Theta_{\lambda+\rho}$ are exactly the characters of the discrete series representations of G. □

Note the close family resemblance between the formula above and the Weyl character formula.

Remark. In the last theorem, one may also realize all the characters of the discrete series by considering only those λ which satisfy $(\lambda + \rho, \alpha) < 0$ for all $\alpha \in \triangle^+$, but then allowing $\triangle^+$ to vary over all the positive root systems of $\triangle$.

The discrete series representations can also be characterized by their homology exponents (§7.1). Let V be a finitely generated admissible $(\mathfrak{g}, K)$-module. Let $\triangle_0 = \{\alpha_1, \ldots, \alpha_r\}$. Define $H_1, \ldots, H_r \in \mathfrak{a}$ by $\alpha_i(H_j) = \delta_{ij}$. Then define $\Lambda_V \in \mathfrak{a}^*$ by

$$\Lambda_V(H_j) = \max\{-\mathrm{Re}\ \mu(H_j) \mid \mu \in \mathcal{A}_{hom}(\hat{V})\}.$$

Theorem 7.2.14 *Consider a Hilbert space representation $\mathcal{H}$ of G with underlying Harish-Chandra module V. Then $\mathcal{H}$ is in the discrete series if and only if $\Lambda_V + \rho \in -(\mathfrak{a}^{*+})$.* □

Since G has discrete series if and only if $\mathrm{rank}(G) = \mathrm{rank}(K)$, we can easily list the non-compact classical groups which have discrete series representations. They are $SU(m,n)$, $SO_o(m,n)$ (one of m or n is even), $Sp(m,n)$, $Sp(n,\mathbb{R})$, and $SO^*(2n)$. Note that none of the complex classical groups are in this list.

7.3 Tempered Representations

Definition 7.3.1 An invariant eigendistribution Θ on G is *bounded at ∞* if for every Cartan subgroup H, $\Theta\big|_H |\delta|$ is bounded.

Consider the Riemannian space G/K and let $\sigma(x)$ denote the distance between the cosets K and xK. For $u \in \mathcal{U}(\mathfrak{g})$ let L_u and R_u denote the differentials of the left and right translation actions on $C^\infty(G)$.

Definition 7.3.2 The *Schwartz space* $\mathcal{S}(G)$ of G is the function space

$$\mathcal{S}(G) = \{f \in C^\infty(G) : (1+\sigma)^m (L_a R_b f) \in L^2(G)\ \forall a, b \in \mathcal{U}(\mathfrak{g}),\ m \in \mathbb{Z}_{\geq 0}\},$$

with topology given by the seminorms $f \mapsto ||(1+\sigma)^m(L_a R_b f)||_2$. A distribution T on G is said to be *tempered* if it extends to a continuous linear functional on $\mathcal{S}(G)$.

The space $C^\infty(G)$ is dense in $\mathcal{S}(G)$ and the inclusion map is continuous. Further $\mathcal{S}(G)$ is a Fréchet space, closed under convolution, and convolution is a continuous operation from $\mathcal{S}(G) \otimes \mathcal{S}(G)$ to $\mathcal{S}(G)$.

Theorem 7.3.3 *If an invariant eigendistribution Θ is bounded at ∞ then it is tempered. If Θ is the global character of a representation then the converse also holds.* □

Definition 7.3.4 An irreducible Hilbert space representation of G is *tempered* if its global character is tempered.

A matrix coefficient $\langle \pi(x)u, v\rangle$ is said to be K-finite if the vectors u and v are smooth and K-finite.

Theorem 7.3.5 (Harish-Chandra, Langlands, Trombi) *Consider an irreducible Hilbert space representation $\mathcal{H}$ of G with underlying Harish-Chandra module V. Then $\mathcal{H}$ is tempered if and only if one of the following holds:*

1. *Every K-finite matrix coefficient of $\mathcal{H}$ is in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$.*
2. *$\Lambda_V + \rho \in -\mathrm{Cl}(\mathfrak{a}^{*+})$.*
3. *There is a standard p-pair (P_F, A_F), an irreducible square integrable representation σ of M_F, and $\mu \in (\mathfrak{a}_F)^*$, such that $\mathcal{H}$ is a summand of $U(P_F, \sigma, i\mu)$.*

□

Consequently, tempered representations are automatically unitary. Note that the last characterization does not quite give a classification of the tempered representations, as it depends on an understanding of the possible summands of parabolically induced representations and the isomorphisms amongst them. But if we recall that a general parabolically induced representation is irreducible, then we realize we are quite close to a classification.

This theorem shows that the discrete series form the building blocks for the tempered representations. The next result shows that the tempered representations, in turn, are the building blocks for general irreducible admissible representations of G.

Consider a standard p-pair (P_F, A_F). Let (σ, H_σ) be a tempered representation of M_F. Let $\mu \in (\mathfrak{a}_F)^*_{\mathbb{C}}$ be such that $\mathrm{Re}(\mu, \alpha) > 0$ for $\alpha \in \triangle(P_F, A_F)$. The triple (P_F, σ, μ) is called *Langlands data*. In this situation, there is a nonzero intertwining operator $j_{F,\sigma,\mu} : U(P_F, \sigma, \mu) \to U(P_F, \sigma, \mu)$ whose image $J_{F,\sigma,\mu}$ is isomorphic to the unique irreducible quotient of $U(P_F, \sigma, \mu)$.

Theorem 7.3.6 (Langlands Classification) *Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then there exists Langlands data (P_F, σ, μ) such that V is $(\mathfrak{g}, K)$-isomorphic with $J_{P_F,\sigma,\mu}$. Further, if V is also isomorphic to $J_{P_{F'},\sigma',\mu'}$ for a Langlands data $(P_{F'}, \sigma', \mu')$, then $F = F'$, $\mu = \mu'$, and σ is unitarily equivalent to σ'* □

This can be seen as a loosening of the subquotient and subrepresentation theorems, in that the irreducible admissible representations are not realized in principal series, but in more general parabolically induced representations.

Example Recall from Chapter 3 the partition of the irreducible $(\mathfrak{g}, K)$-modules of $SL(2,\mathbb{R})$ into the classes $D(\mu, \pm)$, $F(\mu)$, and $P(c; \varepsilon + 2\mathbb{Z})$. In terms of our present classification, the $D(\mu, \pm)$, $\mu \neq 0$, form the discrete series. The mock discrete series $D(0, \pm)$ and the first and second principal series complete the set of tempered representations. Indeed, by Theorem 3.4.2 we have $P(c; \varepsilon + 2\mathbb{Z}) \cong U(B, \varepsilon, \sqrt{c})$ for these principal series. In fact that result gives exactly the realisation of all the irreducibles as Langlands quotients. The non-tempered unitary representations, which do not therefore show up in $L^2(G)$, are the complementary series and the trivial representation $F(1)$.

7.4 Constructions of Discrete Series

We have indicated in the previous section how knowledge of the discrete series is crucial to wider classifications. Now while we have given Harish-Chandra's classification of the discrete series, he classified only their global characters and did not construct the representations themselves in general. The only discrete series we have actually constructed so far are the holomorphic ones. From §7.2 we know that discrete series exist in some cases where G/K is not Hermitian symmetric.[5] And even when G/K is Hermitian symmetric the holomorphic discrete series does not exhaust the full discrete series ($SL(2,\mathbb{R})$ being the lone exception). Since the construction of the holomorphic discrete series is a clear generalization of the Borel-Weil theorem, it is reasonable to try to generalize also the Bott-Borel-Weil theorem to the non-compact case. Thus Kostant [86] and Langlands [94] conjectured that the discrete series could be realized in L^2-cohomology. We give a quick sketch of the conjecture below.

For simplicity we assume that G is linear semisimple, with complexification $G_{\mathbb{C}}$. Fix a compact Cartan subgroup T and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$. Choose $\triangle^+ \subset \triangle(\mathfrak{g},\mathfrak{t})$ so that $\mathfrak{b}$ corresponds to the negative roots. Finally, let $B_{\mathbb{C}}$ be the normalizer of $\mathfrak{b}$ in $G_{\mathbb{C}}$. Now consider the manifold $X = G_{\mathbb{C}}/B_{\mathbb{C}}$. The G-orbit of $\mathfrak{b}$ is open in X and we denote it by D. Then $D \cong G/T$.(See Wolf [153].)

For an integral weight λ of $\mathfrak{t}$, the $G_{\mathbb{C}}$-equivariant line bundle L_λ (see §2.5) on X restricts to a G-equivariant holomorphic line bundle on D. Define

$$p(\lambda+\rho) = \#\{\alpha \in \triangle_c^+;\ (\lambda+\rho,\alpha) < 0\} + \#\{\beta \in \triangle_n^+\ (\lambda+\rho,\alpha) > 0\}.$$

Let $H_2^*(D,\mathcal{O}(L_\lambda))$ be the "separated L^2 cohomology" of the restricted line bundle. Then the conjecture is that $H_2^p(D,\mathcal{O}(L_\lambda)) = 0$ if λ is singular or if $p \neq p(\lambda+\rho)$. If λ is non-singular and $p = p(\lambda+\rho)$ then the conjecture is that $H_2^p(D,\mathcal{O}(L_\lambda))$ is the discrete series representation with character $\Theta_{\lambda+\rho}$.

Narasimhan and Okamoto [108] provided the first confirmation of this conjecture. Working in the Hermitian symmetric context, they obtained most of the discrete series in the above spaces. That is to say, they showed the existence of a constant C such that whenever a $\triangle_c^+$-dominant non-singular integral weight λ also satisfies $|(\lambda+\rho,\alpha)| > C\ \forall\alpha \in \triangle_n^+$, then $H_2^p(D,\mathcal{O}(L_\lambda))$ vanishes if $p \neq p(\lambda+\rho)$, and is the appropriate discrete series representation otherwise. (The case $p(\lambda+\rho) = 0$ corresponds to the holomorphic discrete series.)

[5] Dixmier [19] found the first non-Hermitian symmetric case where the discrete series exists.

Evidence for the conjecture had earlier come from Schmid [120] who showed that the Dolbeault cohomology $H^p(D, \mathcal{O}(L_\lambda))$ vanishes except for $p = p(\lambda + \rho)$, and in that dimension is a Fréchet space which has the K-type multiplicities predicted for the discrete series by the Blattner conjecture. The conjecture was then confirmed for "sufficiently non-singular" λ by Parthasarathy [112] and Schmid [121]. Finally, Zuckerman's method of translating the parameter by tensoring with finite dimensional representations allowed the proof of the full conjecture (Schmid [122]).

The analytic approaches above proved difficult to generalize, for example to realizations of representations through vector bundles of rank > 1.[6] Algebraic analogues (for the construction of the underlying $(\mathfrak{g}, K)$-module) were therefore sought. One success was provided by the Enright and Varadarajan modules [31], and their generalizations by Wallach [143] and Parthasarathy [113].

In the rest of this book we will be concerned with one such algebraic approach, first proposed by Zuckerman and which, after some recasting, has taken the form of "cohomological parabolic induction" or "derived Zuckerman functors". The power of this approach resides in its consisting of two dual families of derived functors. The original ones, due to Zuckerman, are left exact and suitable for settling irreducibility. The others, due to Bernstein, are right exact and suited for unitarity questions. The duality allows us to use whichever ones are convenient.

Knapp and Zuckerman have used these constructions to classify the tempered representations [82, 83, 84]. For instance, suppose we work with the derived Zuckerman functors and use $A_{\mathfrak{b}}(\lambda)$ to denote the underlying $(\mathfrak{g}, K)$-module of the discrete series representation with character $\Theta_{\lambda+\rho}$, where λ is a dominant non-singular integral weight relative to the choice $\triangle^+$ corresponding to the Borel $\mathfrak{b}$. Then the same construction for a dominant but singular λ yields a tempered representation which is either 0 or irreducible, and is 0 if and only if $(\lambda, \alpha) = 0$ for some compact simple root α. These representations are called *limits of discrete series* and the result is that every tempered representation arises by parabolic induction from an appropriate discrete series or a limit of discrete series representation (without recourse to summands).

Integrable representations (see definition 1.5.8) have also been classified. Trombi and Varadarajan [129] gave necessary conditions for a discrete series representation to be integrable, and also sufficient conditions. Later, Hecht and Schmid [57] showed that their necessary conditions were themselves sufficient.

7.5 The Unitary Dual

While the tempered representations solve the main problem as far as harmonic analysis is concerned, from the point of view of representation theory

[6] Examples of analytic methods are Wolf [154], and Rawnsley, Schmid and Wolf [118].

the full unitary dual still remains to be determined in general.[7] A survey of the situation till 1981 can be found in Knapp and Speh [80]. This paper also contains a condition for unitarity involving the intertwining operator used in the Langlands classification. But the condition is hard to verify in general.

Before the early seventies, successes in finding $\widehat{G}$ involved a few classical groups of small size, as well as $SO_0(n,1)$ and $SU(n,1)$. (The last two by Hirai [60, 61, 62, 63], who also calculated the characters.[8]) Advent of the Langlands classification and the derived Zuckerman functors has made possible more general results. Thus the unitary duals of the following groups are now known: $\overline{SO}_0(2n,2)$ (Angelopoulos [1], 1981), $SU(2,2)$ (Knapp and Speh [79], 1982), linear semisimple Lie groups of real rank one with compact Cartan subgroups (Baldoni Silva and Barbasch [4], 1983),[9] $GL(n,F)$ for $F=\mathbb{R},\mathbb{C},\mathbb{H}$ (Vogan [137], 1986), complex classical Lie groups (Barbasch [5], 1989), $\widetilde{GL}(n,\mathbb{R})$ (Huang [67], 1990), and G_2 (Vogan [139], 1994).

Apart from such case-by-case solutions, attempts have also been made to generate large classes of unitary representations across the different group families. We have already described (§6.5) the classification of the irreducible unitary highest weight representations. Another success is the classification of "unitary representations with non-zero cohomology" by Vogan and Zuckerman [140], building upon earlier work of Parthasarathy [113, 114] and Kumaresan [92]. These representations can be realized either through Parthasarathy's generalization of the Enright-Varadarajan modules or through cohomological induction.

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[7] Segal and von Neumann [124] have shown that that the unitary dual of a noncompact connected simple Lie group contains no finite dimensional representations.

[8] Also see Ottoson [110, 111] and Kraljević [88, 89, 90].

[9] The *real rank* or $\mathbb{R}$-*rank* of a real reductive group $G=KAN$ is the real dimension of A.