Chapter 1

Locally Compact Groups

By the 1940's representation theory was well-developed on two fronts. The first concerned locally compact abelian groups, where the theory was a relatively straightforward extension of classical Fourier analysis [35, 116, 150]. The other concerned compact groups. Here, the Peter-Weyl theorem gave the general scheme, and for the special case of compact Lie groups the work of Hermann Weyl and Élie Cartan gave an explicit construction of the irreducible representations. In both these cases the unitary irreducibles are finite dimensional and they can all be obtained by a decomposition of the regular representation $L^2(G)$. When this decomposition is explicitly given, it is called a *Plancherel formula.*

For a general locally compact group the situation is much messier. Murray and von Neumann [105, 106, 107, 141] showed that the unitary representation theory of such a group could be quite intractable. In particular, it need not reduce to the study of the irreducible unitary representations. If it does so reduce, the group is said to be of Type I.

We will list some general facts about locally compact groups and their representations, in order to provide some context to the rest of our story. In this chapter we are particularly skimpy as regards details and proofs. The main theme is the essential role of the "group C^*-algebra". For more on this relationship, and how group theoretic methods open up new prospects in the theory of C^*-algebras, see Dixmier [21] and Doran–Fell [23], respectively.

Let G be a separable, locally compact and Hausdorff topological group. (Henceforth we shall subsume all these properties under the phrase "locally compact group".)

Definition 1.0.1 A *representation* of G is a pair $(\pi, \mathcal{V})$ of a complex vector space $\mathcal{V}$ and a group homomorphism π from G into the invertible linear operators on $\mathcal{V}$. It is called *continuous* if $\mathcal{V}$ is a topological vector space and the map $G \times \mathcal{V} \to \mathcal{V}$, $(g, v) \mapsto \pi(g)(v)$, is continuous.

The representations of G that we consider will be continuous and in a separable Hilbert space. We often invoke a representation $(\pi, \mathcal{V})$ by referring to π or $\mathcal{V}$ alone.[1]

[1]For a finite dimensional vector space V we denote the invertible linear operators by

If $\mathcal{V}$ has a closed subspace $\mathcal{U}$ which is invariant under $\pi(G)$ (i.e. $\pi(g)(\mathcal{U}) \subset \mathcal{U}\ \forall g \in G$), we say $\mathcal{U}$ is a *subrepresentation* of $\mathcal{V}$, with the G action being given by the restrictions of the operators $\pi(g)$ to $\mathcal{U}$. If $\mathcal{U}'$ is another subrepresentation of $\mathcal{V}$ and $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}'$ then we say $\mathcal{U}'$ is *complementary* to $\mathcal{U}$.

Definition 1.0.2 A representation $(\pi, \mathcal{V})$ of G is called *irreducible* if it has no non-zero proper subrepresentations. It is *completely reducible* if for every subrepresentation $\mathcal{U}$ there is another subrepresentation $\mathcal{U}'$ which is complementary to $\mathcal{U}$.

Definition 1.0.3 A representation $(\pi, \mathcal{H})$ of G is called *unitary* if $\mathcal{H}$ is a separable Hilbert space and π maps G into the unitary operators on $\mathcal{H}$.

Given two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of G, an *intertwining operator* from $\mathcal{H}$ into $\mathcal{H}'$ is a bounded operator $T : \mathcal{H} \to \mathcal{H}'$ such that

$$T \circ \pi(x) = \pi'(x) \circ T \qquad \forall x \in G.$$

The space of intertwining operators from $\mathcal{H}$ into $\mathcal{H}'$ is denoted by $R(\pi, \pi')$. In particular $R(\pi) = R(\pi, \pi)$ is an algebra over $\mathbb{C}$, and its center is denoted $CR(\pi)$. Two representations π and π' are *equivalent* if there is a bijective intertwining operator from one to the other. More specifically, two unitary representations π and π' are *unitarily equivalent* if they are equivalent via a unitary operator. We say $\pi \leq \pi'$ if π is unitarily equivalent to a subrepresentation of π', and $\pi \cong \pi'$ if they are unitarily equivalent.

Example 1.0.4 Let $(\pi, \mathcal{H})$ be a unitary representation of G. It is completely reducible: If $\mathcal{U}$ is an invariant closed subspace, then so is $\mathcal{U}^{\perp}$. For any invariant closed subspace $\mathcal{U}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{U}$ is a member of $R(\pi)$. □

Theorem 1.0.5 (Schur's Lemma) *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be Hilbert space representations of G.*

1. *The representation π is irreducible if and only if $R(\pi) = \mathbb{C} \cdot I$.*
2. *If π and π' are both irreducible and $A \in R(\pi, \pi')$ then $A = 0$ or A is an isomorphism and unique up to constants.* □

By the *unitary dual* $\widehat{G}$ we mean the unitary equivalence classes of the irreducible unitary representations of G. (We will usually abuse notation and call an individual representation a member of $\widehat{G}$.)

Recall that a locally compact G has a non-zero Borel measure μ which is finite on compact sets, invariant under left multiplication, and is unique up to positive scalars. This is called a *left Haar measure* of G. For any $x \in G$ define a measure μ_x on G by $\mu_x(A) = \mu(Ax)$, where A is a Borel set. Then this is also a left invariant Borel measure, and so there is a positive real $\triangle_G(x)$ such that $\mu = \triangle_G(x)\mu_x$.[2] The map $\triangle_G$ is a continuous group homomorphism

$GL(V)$. For a Hilbert space $\mathcal{H}$ we denote the bounded linear operators by $\mathcal{B}(\mathcal{H})$.

[2]Some authors define $\triangle_G$ by $\mu_x = \triangle_G(x)\mu$ instead.

into the multiplicative group $\mathbb{R}^\times$ and is called the *modular function* of G. It is independent of the choice of Haar measure. Let us write dg for $d\mu(g)$. If $f \in C_c(G)$, the following identities hold:

$$\begin{aligned}
\int_G f(xg)\,dg &= \int_G f(g)\,dg, \\
\int_G f(gx)\,dg &= \triangle_G(x) \int_G f(g)\,dg, \\
\int_G f(g^{-1})\,dg &= \int_G f(g)\triangle_G(g)\,dg.
\end{aligned}$$

A group G is called *unimodular* if $\triangle_G \equiv 1$, i.e. the left Haar measures are also right invariant. If G is either abelian or compact then it is unimodular.

Example 1.0.6 The examples in this chapter are often summaries of the data available for special cases, rather than an explicit working out of these cases. Of such examples, this one is the first. In it we collect some standard information and notation regarding Lie groups and Lie algebras.

A Lie group is a group G which also has a structure of a finite dimensional second countable differentiable manifold, such that the multiplication and inverse maps are C^∞.[3] Let T_xG denote the tangent space to G at x, viewed as differential operators acting on C^∞ functions defined around x. Each $X \in T_eG$ extends to a left invariant vector field $\tilde{X}$on G by translation:

$$\tilde{X}(x)f = X(f \circ L_x), \qquad \text{where } L_x(y) = xy,\ x, y \in G.$$

This identifies T_eG with the left invariant vector fields, and both these spaces are denoted $\mathfrak{g}$. The second picture for $\mathfrak{g}$ gives it a natural bilinear map into itself, called the Lie bracket, and defined by

$$[X, Y] = X \circ Y - Y \circ X.$$

The Lie bracket is anti-commutative ($[X, Y] = -[Y, X]$) and satisfies the Jacobi identity ($[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$). We sum up these properties of the bracket by calling $\mathfrak{g}$ a Lie algebra. There is a natural map $\exp : \mathfrak{g} \to G$, called the exponential map,[4] which is a local diffeomorphism at $0 \in \mathfrak{g}$ and has the property

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X, \quad \forall X \in \mathfrak{g}.$$

If H is another Lie group, with Lie algebra $\mathfrak{h}$, and $\varphi : G \to H$ is a C^∞ group homomorphism, then the derivative $d\varphi$ at identity is a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathfrak{h}$ and the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
\mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\varphi"] & \mathfrak{h} \arrow[u, "\exp"']
\end{tikzcd}$$

[3] Hence G is a locally compact group.

[4] We also use the notation $\exp(X) = e^X$.

In particular, consider the conjugation map $\sigma_x : G \to G$ defined by an element $x \in G$: $\sigma_x(y) = xyx^{-1}$. Then the derivative of σ_x at identity, $\mathrm{Ad}(x) = d\sigma_x : \mathfrak{g} \to \mathfrak{g}$, is a Lie algebra automorphism. The map $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$ and is called the adjoint representation.[5] Let ad be the derivative of Ad at identity. Then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] & L(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

where $L(\mathfrak{g})$ denotes the space of all linear operators on $\mathfrak{g}$. Moreover, there is a simple formula for ad:

$$\mathrm{ad}(X)(Y) = [X, Y] \qquad \forall X, Y \in \mathfrak{g}.$$

There is a procedure for finding the Haar measure of a Lie group. Let $X_1, \ldots, X_n$ be a basis for T_eG, and let $\tilde{X}_1, \ldots, \tilde{X}_n$ be their extensions to left invariant vector fields on G. For each $x \in G$ the tangent vectors $\tilde{X}_1(x), \ldots, \tilde{X}_n(x)$ form a basis for T_xG, and hence we have a trivialization of the tangent bundle TG. Define an n-form ω on G by

$$\omega(\tilde{X}_1, \ldots, \tilde{X}_n) = 1.$$

Then ω is a left invariant volume form on G. In particular if f is a C^∞ compactly supported function on G then we can define $\int_G f\, dg = \int_G f\omega$ and this will extend to a Haar measure on G.

If G is a Lie group then its modular function is given by

$$\triangle_G(x) = |\det(\mathrm{Ad}(x))|.$$

The details of integration on Lie groups can be found in Helgason [59, Ch I] or Knapp [78, Ch VIII]. □

Example 1.0.7 Let $GL(n, \mathbb{R})$ denote the $n \times n$ invertible matrices with entries in $\mathbb{R}$. If x_{ij} are the coordinates from the matrix entries, then the Haar measure of $GL(n, \mathbb{R})$ is

$$|\det(x_{ij})|^{-n} \prod_{i,j} dx_{ij}.$$

□

Example 1.0.8 Let $SL(n, \mathbb{R})$ denote the $n \times n$ real matrices with determinant equal one. If X_{11} is the cofactor matrix corresponding to the $(1, 1)$ entry, G' consists of those group elements such that $\det(X_{11}) \neq 0$, and x_{ij} are the matrix entries (varying over G'), then the Haar measure of $SL(n, \mathbb{R})$ is

$$|\det(X_{11})|^{-1} \prod_{(i,j)\neq(1,1)} dx_{ij}.$$

□

[5] If G is real then this is a representation in a real vector space.

Example 1.0.9 Consider the Hilbert space $L^2(G, d\mu)$ defined by a left Haar measure of a locally compact group G. Then G acts on it by left translations and these give the *left regular representation* L:

$$(L(x)\phi)(y) = \phi(x^{-1}y), \qquad \forall x, y \in G,\ \phi \in L^2(G, d\mu).$$

The left regular representation is obviously unitary. One may similarly use a right Haar measure $d\nu$ to define the right regular representation R:

$$(R(x)\phi)(y) = \phi(yx), \qquad \forall x, y \in G,\ \phi \in L^2(G, d\nu).$$

If G is unimodular, we can choose $d\mu = d\nu$ and denote the common representation space by $L^2(G)$. Then the left and right regular representations are unitarily equivalent, the equivalence $\Phi : L^2(G) \to L^2(G)$ being given by $(\Phi f)(x) = f(x^{-1})$.

$\square$

We now list some standard constructions of new representations from old.

If $\mathcal{V}$ is a subrepresentation of a representation $(\pi, \mathcal{H})$ then we define the *quotient representation* $(\psi, \mathcal{H}/\mathcal{V})$ by $\psi(g)[u] = [\pi(g)(u)]$ where $[u] = u + \mathcal{V}$ denotes a coset.

If $(\pi_i, \mathcal{H}_i)$, $i = 1, 2, 3, \ldots,$ are representations of G then we define their *direct sum* $(\pi, \mathcal{H})$ by $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, a Hilbert space direct sum, and $\pi(g)|_{\mathcal{H}_i} = \pi_i(g)\ \forall g \in G$.

If $(\pi, \mathcal{H})$ and $(\psi, \mathcal{V})$ are unitary representations of G, then the *tensor product representation* $\pi \otimes \psi$ acts on the completed tensor product $\mathcal{H}\hat{\otimes}\mathcal{V}$ by $(\pi \otimes \psi)(g)(u \otimes v) = \pi(g)(u) \otimes \psi(g)(v)$.[6] A related construction is when $(\pi, \mathcal{H})$ and $(\psi, \mathcal{V})$ are representations of different groups G and H, respectively. Then the *outer tensor product* representation $\pi \otimes \psi$ of the direct product group $G \times H$ on the space $\mathcal{H}\hat{\otimes}\mathcal{V}$ is defined by $(\pi \otimes \psi)(g, h)(u \otimes v) = \pi(g)(u) \otimes \psi(h)(v)$.

The *contragredient representation* $(\pi^c, \mathcal{H}^c)$ of $(\pi, \mathcal{H})$ has representation space $\mathcal{H}^c = \mathcal{H}^*$ and the action is $\pi^c(g) = \pi(g^{-1})^t$, where t means transpose.

The *complex conjugate* representation $\bar{\pi}$ has as representation space $\bar{\mathcal{H}}$, the underlying real vector space of $\mathcal{H}$ equipped with the opposite complex structure. The operator $\bar{\pi}(g)$ is the same as $\pi(g)$ but seen as a complex linear map on $\bar{\mathcal{H}}$.

If the ingredients in the above recipes are unitary then so is the final result. Moreover, if $\mathcal{H}$ is a unitary representation then $\bar{\mathcal{H}}$ and $\mathcal{H}^c$ are unitarily equivalent via the map $u \mapsto \langle \cdot\,, u\rangle$ $(u \in \bar{\mathcal{H}})$.

Example 1.0.10 Let G be a locally compact abelian group. By Schur's lemma every irreducible unitary representation of G is one dimensional. A

[6] Consider the usual tensor product of vector spaces $\mathcal{H} \otimes \mathcal{V}$, and put the inner product $\langle u \otimes v, u' \otimes v'\rangle_\otimes = \langle u, u'\rangle\langle v, v'\rangle$. Now $\mathcal{H} \otimes \mathcal{V}$ has become a pre-Hilbert space and we complete with respect to $\langle\cdot,\cdot\rangle_\otimes$ to get the Hilbert space $\mathcal{H}\hat{\otimes}\mathcal{V}$. Also see Folland [35, §7.3].

$\pi \in \widehat{G}$ corresponds to a continuous homomorphism χ_π from G into the circle group $\mathbb{T}$ (the unit vectors in $\mathbb{C}$ under complex multiplication):

$$\pi(g)(z) = \chi_\pi(g)z, \qquad \forall g \in G, z \in \mathbb{C}.$$

The homomorphism χ_π is called the *character* corresponding to π. $\widehat{G}$ becomes an abelian group under tensor product—this corresponds to point-wise multiplication of characters. It is given the topology of uniform convergence on compact subsets: $\gamma_n \to \gamma$ if $\sup_{x\in C} |\chi_{\gamma_n}(x) - \chi_\gamma(x)| \to 0$ for every compact subset C of G. With this topology $\widehat{G}$ itself is an abelian locally compact group. □

Example 1.0.11 Let $G = \mathbb{T}$, with Haar measure $d\mu = (1/2\pi)d\theta$. The characters are of the form

$$\chi_n(e^{i\theta}) = e^{-in\theta}, \qquad n \in \mathbb{Z}.$$

Let π_n be the representation with character χ_n. This gives an isomorphism of $\widehat{\mathbb{T}}$ with the discrete group $\mathbb{Z}$. The theory of Fourier series can be written in the following form: We have a map, the Fourier transform $f \mapsto \hat{f}$, which maps $L^2(\mathbb{T})$ into ℓ^2 (square summable complex sequences) by

$$\hat{f}(n) = \int_{\mathbb{T}} f\,\chi_n\,d\mu.$$

We have the Fourier inversion formula

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,\bar{\chi}_n,$$

and the Plancherel formula

$$\int_{\mathbb{T}} |f|^2 d\mu = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

The regular representation can therefore be decomposed as:

$$L^2(\mathbb{T}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_n,$$

where $\mathbb{C}_n$ represents the π_n action on $\mathbb{C}$. □

Example 1.0.12 Let $G = \mathbb{R}$ be the additive group with Haar measure $d\mu = (1/\sqrt{2\pi})dx$, where dx is the Lebesgue measure. The characters are

$$\chi_t(x) = e^{-ixt}, \qquad t \in \mathbb{R},$$

and we have $\widehat{\mathbb{R}} \cong \mathbb{R}$ with the usual topology. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(t) = \int_{\mathbb{R}} f\,\chi_t\,d\mu$$

and is an element of $L^2(\mathbb{R})$. We have again an inversion formula

$$f(s) = \int_{\mathbb{R}} \hat{f}\,\bar{\chi}_s\,d\mu,$$

and a Plancherel formula

$$\int_{\mathbb{R}} |f|^2 \, d\mu = \int_{\mathbb{R}} |\hat{f}|^2 \, d\mu.$$

Finally, the Fourier transform extends from the dense subspace $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to an isometry of $L^2(\mathbb{R})$ with itself. The Fourier transform obeys the translation formula: if we define $g(s) = f(s - x)$ for a fixed $x \in \mathbb{R}$ then

$$\hat{g}(t) = e^{-itx} \hat{f}(t).$$

Let π be the representation of $\mathbb{R}$ in $L^2(\mathbb{R})$ given by

$$(\pi(x)g)(t) = e^{-itx} g(t).$$

Then the Fourier transform gives a unitary equivalence of the left regular representation with π. $\square$

Example 1.0.13 The above considerations have been shown to hold for a general locally compact abelian group G. Fix a Haar measure μ on G. For every $f \in L^1(G) \cap L^2(G)$ we can define the corresponding Fourier transform, a function on $\widehat{G}$, by

$$\hat{f}(\chi) = \int_G f \, \chi \, d\mu.$$

There exists a unique Haar measure $\hat{\mu}$ on $\widehat{G}$, called the *dual measure* to μ, such that each $\hat{f}$ is in $L^2(\widehat{G})$ and we have the inversion formula

$$f = \int_{\widehat{G}} \hat{f} \, \bar{\chi} \, d\hat{\mu},$$

as well as the Plancherel formula

$$\int_G |f|^2 \, d\mu = \int_{\widehat{G}} |\hat{f}|^2 \, d\hat{\mu}.$$

The Fourier transform again extends to an isometry of $L^2(G)$ with $L^2(\widehat{G})$. As in the previous example, we have a translation formula: If $g = L(x)f$ then

$$\hat{g}(\chi) = \chi(x)\hat{f}(\chi).$$

So let $\hat{x}$ be the bounded function on $\widehat{G}$ defined by $\hat{x}(\chi) = \chi(x)$, and $M_{\hat{x}}$ the corresponding multiplication operator on $L^2(\widehat{G})$:

$$(M_{\hat{x}}\varphi)(\chi) = \hat{x}(\chi)\varphi(\chi).$$

The Fourier transform gives a unitary equivalence of the left regular representation with the representation $(\pi, L^2(\widehat{G}))$ of G defined by $\pi(x) = M_{\hat{x}}$.

It follows that characters separate points. If x, y are distinct members of G and $\chi(x) = \chi(y)$ for every character χ, then $M_{\hat{x}} = M_{\hat{y}}$. From the above discussion we conclude that $L(x) = L(y)$, which is false since $L^2(G)$ separates points of G.

The map $G \to \widehat{\widehat{G}}$, $x \mapsto \varphi_x$, given by $\varphi_x(\gamma) = \gamma(x)$ is therefore an injective homomorphism of topological groups. *Pontryagin duality* is the assertion that this map is actually an isomorphism. $\square$

Note In the rest of this chapter "representation" means "unitary representation".

1.1 Compact Groups

In this section, we recall some facts about the representation theory of a compact topological group G. This is a simple situation which provides many motivations and tools for later developments. The details of the following claims may be found in various places, one such being Sugiura [127, §I.3].

Let us fix a left Haar measure dg on G. It is automatically finite and right invariant, and we assume it to be normalized so that $\int_G dg = 1$. We have the following basic fact:

Theorem 1.1.1 *Let (π, V) be a finite dimensional representation of a compact group G. Then we can give V an inner product such that π maps G into the unitary operators on V.*

Proof. First equip V with *any* inner product $(\cdot\,,\cdot)$. Now define another inner product $\langle\cdot\,,\cdot\rangle$ on V by

$$\langle u, v\rangle = \int_G (\pi(g)u, \pi(g)v)\, dg.$$

This new inner product is invariant under $\pi(G)$. □

By $\widehat{G}_{\text{fin}}$ we denote the set of finite dimensional members of $\widehat{G}$, and we choose a representative $(\pi_\lambda, \mathcal{H}_\lambda)$ for each equivalence class $\lambda \in \widehat{G}_{\text{fin}}$. Now we fix an orthonormal basis $\{e_i : 1 \le i \le d_\lambda\}$ for each $\mathcal{H}_\lambda$ and define the matrix coefficients

$$\varphi^\lambda_{ij}(g) = \langle \pi_\lambda(g)e_i, e_j\rangle.$$

The φ^λ_{ij} are continuous and hence are in $L^2(G)$. They satisfy the *Schur orthogonality relations*:

$$\langle \varphi^\lambda_{ij}, \varphi^\zeta_{kl}\rangle = \begin{cases} 0 & \lambda \neq \zeta \\ d_\lambda^{-1}\delta_{ik}\delta_{jl} & \lambda = \zeta \end{cases}, \text{where } d_\lambda = \dim \mathcal{H}_\lambda.$$

Thus the matrix coefficients $\{\varphi^\lambda_{ij} : \lambda \in \widehat{G}_{\text{fin}},\ 1 \le i, j \le d_\lambda\}$ form an orthogonal set in $L^2(G)$. Consider the behavior of this set under the right regular action. With $x \in G$, and setting $\pi_\lambda(x)e_i = \sum_k a^k_i e_k$, we get

$$\begin{aligned} (R(x)\varphi^\lambda_{ij})(y) &= \varphi^\lambda_{ij}(yx) \\ &= \langle \pi_\lambda(y)\pi_\lambda(x)e_i, e_j\rangle \\ &= \sum_k a^k_i \langle \pi_\lambda(y)e_k, e_j\rangle \qquad (1.1) \\ &= \sum_k a^k_i \varphi^\lambda_{kj}(y) \end{aligned}$$

Hence the subspace $\mathcal{H}^j_\lambda$ of $L^2(G)$ spanned by $\{\varphi^\lambda_{ij} | 1 \le i \le d_\lambda\}$ is invariant under the right regular representation R. Further, the above calculation shows that it is equivalent to $(\pi_\lambda, \mathcal{H}_\lambda)$ under the bijection $\varphi^\lambda_{ij} \leftrightarrow e_i$. We see therefore that $L^2(G)$ contains a subrepresentation congruent to $\bigoplus_{\lambda\in\widehat{G}_{\text{fin}}} d_\lambda \mathcal{H}_\lambda$. Next we have the famous result that this gives all of $L^2(G)$.

Theorem 1.1.2 (Peter-Weyl [115]) *Let G be a compact group.*

1. *The set $\{\sqrt{d_\lambda}\varphi_{ij}^\lambda\}$, where λ varies over the equivalence classes of finite dimensional irreducible unitary representations of G and $1 \le i, j \le d_\lambda$, forms an orthonormal Hilbert basis for $L^2(G)$.*
2. *Every irreducible unitary representation of G is finite dimensional.*
3. *The right regular representation has the following decomposition:*

$$L^2(G) = \bigoplus_{\lambda\in\widehat{G}} \bigoplus_{i=1}^{d_\lambda} \mathcal{H}_i^\lambda \cong \bigoplus_{\lambda\in\widehat{G}} d_\lambda \mathcal{H}_\lambda. \tag{1.2}$$

□

In the Peter-Weyl theorem, the second claim follows by combining the first claim with the Schur orthogonality relation for inequivalent representations (that relation does not need finite dimensionality). The third claim follows from the first two.

We could similarly decompose the left regular representation L on $L^2(G)$. With the same notation as in (1.1) we find

$$L(x)(\varphi_{ij}^\lambda) = \sum_k \overline{a_j^k}\, \varphi_{ik}^\lambda. \tag{1.3}$$

Hence $\bar{\mathcal{H}}_\lambda^i = \mathrm{span}\{\varphi_{ij}^\lambda | \, 1 \le j \le d_\lambda\}$ is L-invariant, and is equivalent to the complex conjugate representation $\bar{\mathcal{H}}_\lambda$ under the bijection $\varphi_{ij}^\lambda \leftrightarrow e_j$. Consequently it is isomorphic to the contragredient representation $\mathcal{H}_\lambda^c$ under the bijection $\varphi_{ij}^\lambda \leftrightarrow f_j$, where $\{f_j\}$ is the dual basis to $\{e_i\}$.

The left and right regular representations commute and we can define the product representation $R \times L$ of $G \times G$ on $L^2(G)$ by

$$(R \times L)(x, y)f(z) = f(y^{-1}zx).$$

This leads to an identification of $\bigoplus_{i=1}^{d_\lambda} \mathcal{H}_i^\lambda$ and $\mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^c$ as $G \times G$ representations, via the bijection $\varphi_{ij}^\lambda \leftrightarrow e_i \otimes f_j$. Equation (1.2) then leads to the following decomposition:

$$L^2(G) \cong \bigoplus_{\lambda\in\widehat{G}} (\mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^c) \cong \bigoplus_{\lambda\in\widehat{G}} L(\mathcal{H}_\lambda), \tag{1.4}$$

where $L(\mathcal{H}_\lambda)$ denotes the linear operators on $\mathcal{H}_\lambda$. If we let $G \times G$ act on $L(\mathcal{H}_\lambda)$ by $(g, h) \cdot \phi(u) = \pi_\lambda(g)(\phi(\pi_\lambda(h^{-1})u))$, then the second isomorphism is also of $G \times G$ representations. These alternate pictures of the Peter-Weyl decomposition have been included as they are more suitable for generalizing to the non-compact case.

It follows from (1.1) that if $f \in L^2(G)$ then it has the decomposition

$$f = \sum_{\lambda\in\widehat{G}} d_\lambda \sum_{i,j} \langle f, \varphi_{ij}^\lambda\rangle\, \varphi_{ij}^\lambda, \tag{1.5}$$

and this leads to the Parseval equality

$$||f||^2 = \sum_{\lambda \in \widehat{G}} d_\lambda \sum_{i,j} |\langle f, \varphi_{ij}^\lambda \rangle|^2. \tag{1.6}$$

The similarity to ordinary Fourier analysis can be made more explicit. We denote the complex conjugate of an operator $T : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ by T^*. For $f \in L^2(G)$, we define the *Fourier transform* $\hat{f}$ on $\widehat{G}$ by

$$\hat{f}(\lambda) = \int_G f(g)\pi_\lambda(g)^*\, dg.$$

For each $\lambda \in \widehat{G}$, $\hat{f}(\lambda)$ is an operator on $\mathcal{H}_\lambda$.[7] The Parseval equality can now be written as

$$||f||^2 = \sum_{\lambda \in \widehat{G}} d_\lambda \mathrm{Tr}\,(\hat{f}(\lambda)\hat{f}(\lambda)^*).$$

The methods that lead to Schur's lemma and the Peter-Weyl theorem (spectral analysis of self-adjoint operators) lead also to the following result:

Theorem 1.1.3 *Let $\mathcal{H}$ be a unitary representation of G. Then it is a direct sum of irreducible unitary representations of G.* □

For $\lambda \in \widehat{G}$, we define the *character* Θ_λ by $\Theta_\lambda(g) = \mathrm{Tr}\,\pi_\lambda(g)$, $g \in G$. Then the Schur orthogonality relations for the matrix coefficients imply that for $\lambda, \zeta \in \widehat{G}$,

$$\langle \Theta_\lambda, \Theta_\zeta \rangle = \begin{cases} 0 & \lambda \neq \zeta \\ 1 & \lambda = \zeta \end{cases},$$

where we have the L^2 inner product on the left. Together with the previous theorem this classifies the finite dimensional unitary representations of G by their characters. In fact the characters give us a way of calculating the decomposition of an arbitrary unitary representation $(\pi, \mathcal{H})$ into its irreducible constituents.

Definition 1.1.4 If $\mathcal{H}$ is a unitary representation of G and $\lambda \in \widehat{G}$, the *isotype* $\mathcal{H}(\lambda)$ is the union of all the copies of $\mathcal{H}_\lambda$ in $\mathcal{H}$. It is a subrepresentation of $\mathcal{H}$.

Example 1.1.5 Consider the right regular representation. By (1.5) the orthogonal projection $L^2(G) \to L^2(G)(\lambda)$ is given by

$$f \mapsto d_\lambda \sum_{i,j} \langle f, \phi_{ij}^\lambda \rangle \phi_{ij}^\lambda.$$

[7] Given a locally compact group G and a Hilbert space $\mathcal{H}$, we say a function $f : G \to \mathcal{H}$ is measurable if the functions $x \mapsto \langle f(x), u \rangle$ are measurable for each choice of $u \in \mathcal{H}$. A measurable f is L^1 if $\int_G ||f(x)||\, dx < \infty$. If f is L^1 we define the vector $\int_G f(x)\, dx$ by the equation $\langle \int_G f(x)\, dx, u \rangle = \int_G \langle f(x), u \rangle\, dx \ \forall u \in \mathcal{H}$. Continuing in this vein, the Fourier transform is defined by $\langle \hat{f}(\lambda)u, v \rangle = \int_G f(g) \langle \pi_\lambda(g)^* u, v \rangle\, dg \ \forall u, v \in \mathcal{H}$.

We can express the right side as follows:

$$\begin{aligned} d_\lambda \sum_{i,j} \langle f, \phi_{ij}^\lambda \rangle \phi_{ij}^\lambda(x) &= d_\lambda \sum_{i,j} \int_G f(g) \overline{\phi_{ij}^\lambda(g)} \phi_{ij}^\lambda(x)\, dg \\ &= d_\lambda \sum_{i,j} \int_G f(g) \overline{\phi_{ji}^\lambda(x^{-1}) \phi_{ij}^\lambda(g)}\, dg \\ &= d_\lambda \sum_{j} \int_G f(g) \overline{\phi_{jj}^\lambda(x^{-1}g)}\, dg \\ &= d_\lambda \sum_{j} \int_G f(xg) \overline{\phi_{jj}^\lambda(g)}\, dg \\ &= d_\lambda \int_G \overline{\Theta_\lambda(g)} (R(g)f)(x)\, dg. \end{aligned}$$

□

Now we define

$$\chi_\lambda = d_\lambda \Theta_\lambda, \qquad \lambda \in \widehat{G}.$$

Theorem 1.1.6 *If $\mathcal{H}$ is a unitary representation of G and $\lambda \in \widehat{G}$, then $P(\lambda) : \mathcal{H} \to \mathcal{H}$ defined by*

$$P(\lambda)w = \int_G \overline{\chi_\lambda(g)} \pi(g)(w)\, dg$$

is an orthogonal projection onto the isotype $\mathcal{H}(\lambda)$ of $\mathcal{H}$. □

1.2 The Group C^*-algebra

See §A.1 for the definitions involving $*$-algebras and their representations.

Let us return to the general locally compact case. Consider $L^1(G)$ with the convolution product

$$(f * g)(t) = \int_G f(s) g(s^{-1}t)\, ds$$

and the involution

$$f^*(x) = \triangle_G(x) \overline{f(x^{-1})}.$$

This gives $L^1(G)$ the structure of a Banach $*$-algebra. If G is discrete then the Dirac delta function is a member of $L^1(G)$ and hence the $*$-algebra has an identity. In general, it has only an approximate identity.

Representations of G are in correspondence with $*$-representations of $L^1(G)$ in the following manner: If $(\pi, \mathcal{H})$ is a unitary representation of G, we obtain a non-degenerate $*$-representation of $L^1(G)$ in $\mathcal{H}$, also denoted π, by defining

$$\pi(f)v = \int_G f(x)\pi(x)(v)\, dx \qquad \forall f \in L^1(G), v \in \mathcal{H}.$$

Note that $\pi(L(x)f) = \pi(x)\pi(f)$. On the other hand, suppose we have a $*$-representation $(\pi, \mathcal{H})$ of $L^1(G)$. Then $M = \operatorname{span}\{\pi(f)v : f \in L^1(G),\ v \in \mathcal{H}\}$

is dense in $\mathcal{H}$. For $x \in G$ define the action on M by $\pi(x)(\pi(f)v) = \pi(L(x)f)v$, and then extend $\pi(x)$ to $\mathcal{H}$ by continuity. This establishes a bijection between unitary representations of G and non-degenerate $*$-representations of $L^1(G)$.

Example 1.2.1 Consider the left regular representation $(L, L^2(G))$ of G. Let $f \in L^1(G)$. Then $L(f) : L^2(G) \to L^2(G)$ is given by

$$\begin{aligned}(L(f)(F))(y) &= \int_G f(x)(L(x)F)(y)\, dx \\ &= \int_G f(x)F(x^{-1}y)\, dx \\ &= (f * F)(y),\end{aligned}$$

i.e. $L(f)(F) = f * F$ for every $F \in L^2(G)$.[8] The representation L of $L^1(G)$ is injective: $L(f) = 0$ implies in particular that $f * F = 0$ for every continuous function F with compact support, and hence $f = 0$. □

Definition 1.2.2 The *group C^*-algebra* of G, denoted $C^*(G)$, is the enveloping C^*-algebra of $L^1(G)$.[9]

The correspondence described above between group and algebra representations gives a bijection of the equivalence classes of unitary representations of G with equivalence classes of non-degenerate $*$-representations of $C^*(G)$. This restricts to a bijection of $\widehat{G}$ with $\widehat{C^*(G)}$, and we can transfer to $\widehat{G}$ the *Fell topology* of $\widehat{C^*(G)}$. For abelian G this is the same as the usual topology of compact convergence on $\widehat{G}$. For compact G this gives $\widehat{G}$ the discrete topology. But in general it is not even T_0.[10]

Example 1.2.3 For a compact group G, the Peter-Weyl theory can be recast to give the structure of $C^*(G)$. Let us write $\mathcal{M}_\gamma$ for the C^*-algebra of complex $d_\gamma \times d_\gamma$ matrices, where $\gamma \in \widehat{G}$. Then we have an isomorphism

$$C^*(G) \cong \oplus_{\gamma \in \widehat{G}} \mathcal{M}_\gamma,$$

where the right hand side is a direct sum of C^*-algebras (see §A.1 and also compare with equation (1.2)). □

Example 1.2.4 Suppose G is a locally compact abelian group. Then $L^1(G)$ is abelian and hence also $C^*(G)$. From the Gelfand-Naimark theorem (Theorem A.1.5) we have $C^*(G) \cong C_0(\widehat{C^*(G)}) \cong C_0(\widehat{G})$. What we have skipped here is the proof that the bijections respect the topologies. □

We can give $\widehat{G}$ the Borel structure from the Fell topology, which we will call the Fell Borel structure. There is another, more obvious, Borel structure which we now describe. Fix Hilbert spaces $\mathcal{H}_n$, $n = 1, 2, \ldots, \infty$, with

[8] The integral $\int_G f(x)F(x^{-1}y)\, dx$ converges absolutely for almost every x and $||f*F||_2 \leq ||f||_1||F||_2$ ([35, Proposition 2.39]).

[9] In this case, because of the injective representation in the preceding example, the semi-norm $||\cdot||'$ is a norm and $L^1(G)$ embeds densely in $C^*(G)$.

[10] A topology is T_0 if for every pair of distinct points there is an open set containing one but not the other.

$\dim \mathcal{H}_n = n$. Let $\tilde{G}^n$ be the set of irreducible unitary representations of G on $\mathcal{H}_n$ (*not* their equivalence classes). Let $\tilde{G} = \cup_n \tilde{G}^n$. Let $\mathcal{B}_n$ be the smallest σ-algebra on $\tilde{G}^n$ such that all the matrix coefficients of all the $\pi \in \tilde{G}^n$ are measurable. Let $\mathcal{B}$ be the σ-algebra on $\tilde{G}$ defined by: $E \in \mathcal{B}$ if and only if $E \cap \tilde{G}^n \in \mathcal{B}_n \ \forall n$. Give $\widehat{G}$ the Borel structure $\mathcal{M}$ that it inherits as a quotient of $\tilde{G}$. This is called the *Mackey Borel structure*.

The Mackey Borel structure is in general strictly larger than the Fell Borel structure. In particular it always distinguishes points, as each singleton is in $\mathcal{M}$ (Mackey [100, Thm 8.4]). But the Fell Borel structure only distinguishes points if the map $\pi \mapsto \ker \pi$ from $\widehat{G}$ into the ideal space of $C^*(G)$ is injective.

1.3 Type I Representations and Groups

Mackey [101] is our general reference for this section and others are provided as needed. It will be useful to keep the following example in mind:

Example 1.3.1 Let G be the additive group $\mathbb{R}$ with the following representation in $L^2(\mathbb{R})$:

$$(\pi(t)f)(x) = e^{-itx} f(x).$$

This representation is highly reducible: given any Lebesgue measurable subset of $\mathbb{R}$, the functions supported on it form a subrepresentation. On the other hand, the irreducible unitary representations of G are given by the characters $\chi_t(x) = e^{-itx}$ $(t \in \mathbb{R})$, and it is easily seen that none of these are subrepresentations of π. □

This example shows that direct sum decompositions are inadequate for reducing the study of general unitary representations to that of irreducible ones. Nevertheless, the following consequence of Schur's lemma provides the key definitions for the general situation.

Theorem 1.3.2 *Suppose $(\pi_i, \mathcal{H}_i)$, $i = 1, 2, \ldots$, are irreducible unitary representations of a locally compact group G and $\pi = \oplus_i \pi_i$. Then*
1. *$CR(\pi)$ is one dimensional if and only if $\pi_i \cong \pi_j \ \forall i, j$.*
2. *$R(\pi)$ is abelian if and only if $\pi_i \ncong \pi_j \ \forall i, j$.* □

Definition 1.3.3 A representation π of G is *primary* if $CR(\pi)$ is one dimensional. It is *multiplicity free* if $R(\pi)$ is abelian. It follows that if π is multiplicity free and is a direct sum of primary representations π_i, then all the π_i are irreducible and mutually inequivalent.

Naturally, we ask if Theorem 1.3.2 gives all the representations of the above types. The next example shows it doesn't give all the multiplicity free representations.

Example 1.3.4 Consider Example 1.3.1. Let $\mathcal{H} = L^2(\mathbb{R})$ and $T : \mathcal{H} \to \mathcal{H}$ be an intertwining operator for the representation $(\pi, \mathcal{H})$. For $t \in \mathbb{R}$ let M_t be the multiplication operator on $L^2(\mathbb{R})$ given by the character χ_t: $M_t f(x) = \chi_t(x) f(x)$. Then $TM_t = M_t T$ for every t and hence T itself is given as multiplication by some bounded Borel function [119, p. 325]. It follows that

$R(\pi)$ is abelian and π is multiplicity free. The intuitive justification for this is that one might think of $(\pi, \mathcal{H})$ as being an action on the L^2 sections of a bundle for which each fibre is $\mathbb{C}$, the base space is $\mathbb{R}$, and on the fibre over t the action is by the character χ_t. This idea is made more precise in the next section. $\square$

For primary representations, we have the following:

Theorem 1.3.5 *If $(\pi, \mathcal{H})$ is a primary representation of G and has an irreducible subrepresentation $\mathcal{V}$ then $\mathcal{H} = n\mathcal{V}$ for some $n = 1, 2, \ldots, \infty$.* $\square$

Definition 1.3.6 A primary representation is of *Type I* if it has an irreducible subrepresentation.[11]

Definition 1.3.7 We say G is *Type I* if each of its primary representations is Type I.

Clearly abelian groups are Type I, as are compact groups. Reductive Lie groups are also Type I, as we shall see later.[12]

Consider again the topology and Borel structures on $\widehat{G}$. Let us fix some terminology. A family $\mathcal{F}$ of Borel sets of some Borel space X is a separating family if $x \neq y$ implies the existence of a member A of $\mathcal{F}$ such that $x \in A$ and $y \notin A$. X is *countably separated* if there is a countable separating family of Borel sets. X is *standard* if it is isomorphic to a Borel subspace of some complete separable metric space. There is the following surprising result:

Theorem 1.3.8 (Kuratowski) *Every standard Borel space is isomorphic as a measure space to the unit interval $[0, 1]$ with the usual Borel structure, or to a countable set whose Borel structure consists of all subsets.*

Proof. [126, p. 99] $\square$

It turns out that the Borel structure on $\widehat{G}$ is, in the words of Mackey, either "very good" or "very bad", and the two cases are precisely separated by the property of being Type I. For we have:

Theorem 1.3.9 *The Mackey Borel structure on $\widehat{G}$ is either standard or not even countably separated. Further, the following are equivalent:*

1. *G is Type I.*
2. *The Fell topology is T_0.*
3. *The map $\pi \mapsto \ker \pi$, from $\widehat{G}$ into the set of ideals of $C^*(G)$, is injective.*
4. *The Mackey Borel structure is standard.*
5. *The Mackey Borel structure is the same as the Fell Borel structure.*
6. *For each $\pi \in \widehat{G}$, $\pi(C^*(G))$ contains all the compact operators on $\mathcal{H}_\pi$.*

[11] In terms of the terminology of von Neumann algebras (§A.1), we have that a representation is primary if the von Neumann algebra A generated by it is a factor. It is primary of Type I if A is a factor of Type I. Hence primary representations are also called *factor* representations.

[12] Nilpotent Lie groups are also Type I. However, there are non Type I connected Lie groups. The *Mautner group* is a five dimensional solvable Lie group which is not Type I [2]. Dixmier [20] has exhibited a Type I Lie group whose universal covering group is not Type I.

□

The equivalences are due partly to Dixmier and Fell, but mostly to Glimm [41]. The theorem indicates that Type I groups are the only ones for whom we can hope to understand the structure of $\widehat{G}$.

Definition 1.3.10 A C^*-algebra $\mathcal{A}$ is called CCR[13] (or *liminaire*) if for every irreducible $*$-representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, $\pi(\mathcal{A})$ is contained in the space of compact operators on $\mathcal{H}$. A group G is CCR if $C^*(G)$ is CCR.

Theorem 1.3.11 *G is CCR if and only if the Fell topology of $\widehat{G}$ is T_1.*[14] *(Hence G is Type I.)*

Proof. The "only if" part of the result is a consequence of Corollary A.2.3. □

The theory of direct integrals that we discuss in the next section shows that if a group is Type I then its representation theory has a relatively simple structure. In particular, the irreducible representations suffice for the study of general representations. Let us set up some definitions in anticipation of these results.

Definition 1.3.12 Let π and π' be representations of G. They are *disjoint* if they have no common subrepresentations. We say $\pi \prec \pi'$ if no subrepresentation of π is disjoint from π'. The two are *quasi-equivalent*, denoted $\pi \sim \pi'$, if $\pi \prec \pi'$ and $\pi' \prec \pi$.

If π is primary and $\pi \sim \pi'$ then π' is also primary. Further, a quasi-equivalence class of primary representations is minimal with respect to the ordering $\prec$.

Definition 1.3.13 A representation π is said to be *Type I* if $\pi \sim \pi'$ for some multiplicity free π'.

Theorem 1.3.14 *If $(\pi, \mathcal{H})$ is a Type I representation, then there are unique projections $E_\infty, E_1, E_2, \ldots$ in $CR(\pi)$ such that $\sum_{i=1}^{\infty} E_i = 1$, $E_iE_j = 0$ if $i \neq j$, $E_j(\mathcal{H}) = j \cdot W_j$ for some multiplicity free W_j. Thus*

$$\mathcal{H} = \infty \cdot W_\infty \oplus 1 \cdot W_1 \oplus 2 \cdot W_2 \oplus \cdots$$

□

Theorem 1.3.15 *If G is a Type I group, then all its unitary representations are Type I.* □

[13] CCR stands for *completely continuous representations*, "completely continuous" being another name for a compact operator.

[14] A topology is T_1 if points are closed.

1.4 Direct Integrals

The notion of a direct integral of Hilbert spaces generalizes that of direct sum.[15] Recall that a direct sum $\oplus_{i=1}^{\infty}\mathcal{H}_i$ of Hilbert spaces consists of ∞-tuples (x_i) such that $x_i \in \mathcal{H}_i$ and $\sum_{i=1}^{\infty} ||x_i||^2 < \infty$.

We start with a locally compact space Z, a positive Borel measure μ on Z, and a family $\{\mathcal{H}(z)\}$ of separable Hilbert spaces indexed by the points of Z. By a *vector field* X we mean an assignation of a vector $X(z) \in \mathcal{H}(z)$ for each $z \in Z$. Similarly, an *operator field* A is an assignation to each z of a bounded operator $A(z)$ on $\mathcal{H}(z)$. Then the operator fields act on the vector fields by $(AX)(z) = A(z)X(z)$.

If we try to imitate the direct sum construction, it is natural to try to define the direct integral space $\mathcal{H} = \int_Z \mathcal{H}(z)\, d\mu(z)$ as consisting of vector fields X which satisfy $\int_Z ||X(z)||^2\, d\mu(z) < \infty$, and then to equip $\mathcal{H}$ with the inner product $(X, Y) = \int_Z \langle X(z), Y(z)\rangle\, d\mu(z)$. To do this, we need to impose some conditions of measurability on our vector fields. We start by assuming given a *fundamental family* Λ of vector fields. This means that the following hold:

1. Λ is countable, and for each $z \in Z$ the span of $\{X(z) : X \in \Lambda\}$ is dense in $\mathcal{H}(z)$.
2. The map $\langle X, Y\rangle :\ z \mapsto \langle X(z), Y(z)\rangle$ is measurable $\forall X, Y \in \Lambda$.

A vector field X is called measurable if $\langle X, Y\rangle$ is measurable for each $Y \in \Lambda$. If X, Y are measurable vector fields, then also the function $\langle X, Y\rangle$ is measurable. The direct integral space $\int_Z \mathcal{H}(z)\, d\mu(z)$ can now be defined: It consists of measurable vector fields X such that $\int_Z ||X(\zeta)||^2 d\mu(\zeta) < \infty$. It becomes a Hilbert space with the inner product

$$(X, Y) = \int_Z \langle X(z), Y(z)\rangle d\mu(z).$$

An operator field A is called measurable if its matrix coefficients $\langle AX, Y\rangle$ are measurable functions $\forall X, Y \in \Lambda$. Suppose that A is measurable and $||A||_\infty = \text{ess sup}_{z\in Z} ||A(z)|| < \infty$. Then A is a bounded operator on the space $\int_Z \mathcal{H}(z)\, d\mu(z)$, with norm $||A||_\infty$. We denote A by $\int_Z A(z)\, d\mu(z)$.

To construct a representation of G in a direct integral space we start with Z, Λ, μ, and $\{\mathcal{H}(z)\}_{z\in Z}$ as above, and suppose also a unitary representation π_z of G in each $\mathcal{H}(z)$. If each operator field $\pi_z(x)$, $x \in G$, is measurable, then we get a unitary representation of G on $\int_Z \mathcal{H}(z)\, d\mu(z)$, and we denote it by $\int_Z \pi_z\, d\mu(z)$.

Example 1.4.1 Consider the representation $(\pi, \mathcal{H})$ of Examples 1.3.1 and 1.3.4. We can now see that $\pi = \int_{\mathbb{R}} \chi_t\, d\mu(t)$ where μ is Lebesgue measure, and the fundamental family consists of the vector field which is identically 1. In fact, considering the more general case outlined in Example 1.0.13, we see that for a locally compact abelian group G the left regular representation L

[15] This notion of direct integral or "continuous sum" was introduced by von Neumann in 1938 [142]. Our references are Bruhat [15] and Folland [35].

on $L^2(G, \mu)$ decomposes as

$$L \cong \int_{\widehat{G}} \chi_\lambda \, d\hat{\mu}(\lambda),$$

where μ is a Haar measure on G and $\hat{\mu}$ is the corresponding dual measure on $\widehat{G}$. $\square$

Now we consider the ways in which an arbitrary unitary representation $(\pi, \mathcal{H})$ of G may decompose into a direct integral of unitary representations.

Example 1.4.2 Let $(\pi, \mathcal{H}) = (\oplus\pi_i, \oplus\mathcal{H}_i)$ be a direct sum of unitary representations of G. Then the projections onto invariant subspaces are in $R(\pi)$. Let us consider their linear combinations. For every $\lambda = (\lambda_i)_{i=1}^{\infty} \in l^\infty$ (i.e. $\sup_i |\lambda_i| < \infty$) we have an operator $M_\lambda : \mathcal{H} \to \mathcal{H}$ defined by $M_\lambda|_{\mathcal{H}_i} = \lambda_i$. Then $\mathcal{M} = \{M_\lambda : \ \lambda \in l^\infty\}$ is a commutative weakly closed $*$-subalgebra of $R(\pi)$.

Suppose each π_i is irreducible. Then $\mathcal{M}$ is maximal among the commutative weakly closed $*$-subalgebras of $R(\pi)$. For if $TM_\lambda = M_\lambda T \ \forall \lambda \in l^\infty$, then $T : \mathcal{H}_i \to \mathcal{H}_i$, and hence $T|_{\mathcal{H}_i} = \lambda_i$ for some $\lambda_i \in \mathbb{C}$ (by Schur's lemma). If T is bounded so is $\lambda = (\lambda_i)$.

Conversely, let $\mathcal{M}$ be maximal. Let $T \in R(\pi_n)$. Define $\tilde{T} : \mathcal{H} \to \mathcal{H}$ by $\tilde{T} = 1 \oplus \cdots \oplus 1 \oplus T \oplus 1 \oplus \cdots$. Then it is easy to check that $\tilde{T}$ is in $R(\pi)$ and commutes with $\mathcal{M}$. Hence $\tilde{T} = M_\lambda$ for some $\lambda \in l^\infty$, and $T = \lambda_n$. Therefore $R(\pi_n)$ is 1 dimensional. From Schur's lemma it follows that each π_n is irreducible. $\square$

Now, suppose we have a direct integral $\mathcal{H} = \int_Z \mathcal{H}(z) \, d\mu(z)$. For every $f \in L^\infty(Z, \mu)$ we define a *diagonal operator* $M_f : \mathcal{H} \to \mathcal{H}$ by

$$(M_f X)(z) = f(z)X(z).$$

Then $M_f \in \mathcal{B}(\mathcal{H})$ with $||M_f|| = ||f||_\infty$. If $\pi = \int_Z \pi_z \, d\mu(z)$ is a representation of G on $\mathcal{H}$ then $\mathcal{M} = \{M_f : f \in L^\infty(Z, \mu)\}$ is a commutative weakly closed $*$-subalgebra of $R(\pi)$.

Theorem 1.4.3 *Let $(\pi, \mathcal{H})$ be a unitary representation of G, and $\mathcal{M}$ a commutative weakly closed $*$-subalgebra of $R(\pi)$. Then there is a direct integral decomposition $\int_Z \pi_z \, d\mu(z)$ of π on a space $\int_Z \mathcal{H}(z) \, d\mu(z)$ where Z is a standard Borel space, such that the isomorphism maps $\mathcal{M}$ to the diagonal operators on $\int_Z \mathcal{H}(z) \, d\mu(z)$.* $\square$

There are two important cases:

Theorem 1.4.4 *Consider the situation of the above theorem.*

1. *(Mautner [102]) $\mathcal{M}$ is maximal among the commutative weakly closed $*$-subalgebras of $R(\pi)$ if and only if almost all the π_z are irreducible.*
2. *(von Neumann [142]) If $\mathcal{M} = CR(\pi)$ then almost all the π_z are primary. This is called the* canonical decomposition *of $\mathcal{H}$.* $\square$

The result of Mautner implies that G has many irreducible representations:

Theorem 1.4.5 (Gelfand-Raikov [39]) *Let G be a locally compact group and $g_1, g_2 \in G$, $g_1 \neq g_2$. Then G has an irreducible unitary representation ψ such that $\psi(g_1) \neq \psi(g_2)$.*

Proof. Consider a decomposition into irreducibles of the left regular representation of G: $L^2(G) = \int_Z \mathcal{H}(z)\, d\mu(z)$, and let $L(g) = \int_Z L_z(g)\, d\mu(z))$ be the corresponding decompositions of the operators. If $L_z(g_1) = L_z(g_2)$ for every $z \in Z$ then we reach the incorrect conclusion that $L(g_1) = L(g_2)$. □

To classify all unitary representations of G in terms of the members of $\widehat{G}$ we need some sort of uniqueness of the direct integral decompositions. The first choice involved is of the commutative subalgebra $\mathcal{M}$, and the decomposition depends essentially on it. For instance, starting with two distinct maximal $\mathcal{M}$, one can construct examples where π has two direct integral decompositions into irreducibles, with no irreducible components in common![16] Type I groups provide an exception. For a start, Theorems 1.3.14 and 1.3.15 tell us we only have to classify the multiplicity free representations, and for these $R(\pi)$ is abelian, so there is only one choice of $\mathcal{M}$.

Furthermore, even for a given choice of $\mathcal{M}$, the existence theorem does not give any uniqueness for the measure space Z or the measure μ. However, for Type I groups one can get uniqueness by insisting on $Z = \widehat{G}$.

By a theorem of Mackey [100, Thm 10.2], $\widehat{G}$ being standard means there is a choice of representatives $\pi_\lambda \in \lambda$ of the members of $\widehat{G}$ such that the operator fields defined by the G actions are measurable. Recall that two Borel measures are in the same *measure class* if they have the same measure zero sets. Let C be a Borel measure class on $\widehat{G}$. Let μ be a member. Define $\pi^C = \int_{\widehat{G}} \pi_\lambda\, d\mu(\lambda)$.

Conversely, given a multiplicity free $\pi = \int_Z \pi_z\, d\mu(z)$ of a Type I group G, let $\phi : Z \to \widehat{G}$ be the map $z \mapsto [\pi_z]$. The map ϕ is measurable and hence gives a measure μ_π on the Borel sets of $\widehat{G}$. Its measure class C_π is independent of Z and μ.

Theorem 1.4.6 *Let G be Type I. Then the multiplicity free representations of G are in bijection with the Borel measure classes on $\widehat{G}$, and the bijection is given by $\pi \mapsto C_\pi$, $C \mapsto \pi^C$. Further, $\pi \prec \pi'$ if and only if C_π is absolutely continuous with respect to $C_{\pi'}$.* □

1.5 The Regular Representation

We devote this section to a basic problem of harmonic analysis: the decomposition of the regular representation on $L^2(G)$ of a Type I unimodular locally compact group G.[17] Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then a

[16] An example is in §3.5 of Mackey [101].

[17] Recall that for a unimodular group the left and right regular representations are equivalent.

matrix coefficient $\varphi_{u,v}$ of π is a map from G into $\mathbb{C}$ defined by

$$\varphi_{u,v}(x) = \langle \pi(x)u, v\rangle, \quad u, v \in \mathcal{H}.$$

Theorem 1.5.1 *Let $(\pi, \mathcal{H}) \in \widehat{G}$. If any matrix coefficient of π is in $L^p(G)$, for $p \geq 1$, then there are dense subsets U and V of $\mathcal{H}$ such that $\varphi_{u,v} \in L^p(G)$ for every $(u, v) \in U \times V$.*

Proof. Let $x, y \in \mathcal{H}$ such that $\varphi_{x,y} \in L^p(G)$. Then the vectors given by finite linear combinations $\sum_i a_i \pi(g_i)(x)$, where the a_i are scalars, are dense in $\mathcal{H}$ due to the irreducibility of $\mathcal{H}$. Similarly, the expressions $\sum_j b_j \pi(h_j)(y)$ are also dense. Let $u = \sum_i a_i \pi(g_i)(x)$ and $v = \sum_j b_j \pi(h_j)(y)$. Then

$$\begin{aligned} \varphi_{u,v}(g) &= \sum_{i,j} a_i \bar{b}_j \langle \pi(gg_i)(x), \pi(h_j)(y)\rangle \\ &= \sum_{i,j} a_i \bar{b}_j \langle \pi(h_j^{-1} g g_i)(x), y\rangle \\ &= \sum_{i,j} a_i \bar{b}_j L(h_j) R(g_i) \varphi_{x,y}(g), \end{aligned}$$

where L and R are left and right translations. Thus $\varphi_{u,v}$ is a finite linear combination of translates of $\varphi_{x,y}$, and hence is also in $L^p(G)$. □

For $p = 2$ we have a sharper result:

Theorem 1.5.2 (Godement [42]) *Let $\pi \in \widehat{G}$. Then the following are equivalent:*

1. *π is a direct summand of the regular representation.*
2. *Some non-zero matrix coefficient of π is in $L^2(G)$.*
3. *All matrix coefficients of π are in $L^2(G)$.* □

An irreducible unitary representation that satisfies these conditions is said to be *square integrable*, or to be a member of the *discrete series*. The square integrable representations satisfy the Schur orthogonality relations:

Theorem 1.5.3 (Godement [42]) *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be square integrable representations of G. Let φ and φ' denote their matrix coefficients. Then, writing $\langle \cdot, \cdot \rangle_2$ for the L^2 inner product, we have*

1. *There is a constant d_π (depending on the Haar measure), which satisfies*

$$\langle \varphi_{u,v}, \varphi_{u',v'}\rangle_2 = d_\pi^{-1} \langle u, u'\rangle \langle v, v'\rangle \quad \forall u, u', v, v' \in \mathcal{H}.$$

The constant d_π is called the formal degree *of π.*

2. *If π and π' are not unitarily equivalent, then*

$$\langle \varphi_{u,v}, \varphi'_{u',v'}\rangle_2 = 0 \quad \forall u, v \in \mathcal{H},\ u', v' \in \mathcal{H}'.$$

□

The similarity of square integrable representations with the compact group case goes further:

Theorem 1.5.4 *Let $(\pi, \mathcal{H}_\pi) \in \widehat{G}$ be square-integrable and $\Phi : L^2(G) \to \mathcal{H}_\pi$ the associated intertwining projection. If $f \in L^1(G) \cap L^2(G)$ then $\pi(f)$ is Hilbert-Schmidt (so that $\pi(f)\pi(f)^*$ is trace class)*[18] *and*

$$\int_G |\Phi f(x)|^2 dx = d_\pi \mathrm{Tr}\,(\pi(f)\pi(f)^*).$$

Proof. [21, p315] □

In the non-compact case, the square integrable representations do not exhaust $\widehat{G}$. (For instance, it is obvious that the trivial representation is not square integrable. It is almost as obvious that any finite dimensional unitary representation is not square integrable.) Nor do they exhaust $L^2(G)$. The part that is left over will have a direct integral decomposition.

Definition 1.5.5 The *support*, $\mathrm{supp}(\pi)$, of a unitary representation π is the set of members of $\widehat{G}$ which are weakly contained in π. The support of the regular representation is called the *reduced dual* $\widehat{G}^r$.

Theorem 1.5.6 (Takenouchi) *The equality $\widehat{G}^r = \widehat{G}$ holds if and only if the trivial one-dimensional representation is in $\widehat{G}^r$.* □

Theorem 1.5.7 *An irreducible unitary representation of G is in the reduced dual if it has a diagonal matrix coefficient which is in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.*[19]

□

As a generalization of commutative Fourier analysis, as well as the compact group case, there is the following *Plancherel theorem*:

Theorem 1.5.8 (Segal [123]) *There is a unique positive Borel measure μ on $\widehat{G}$ such that, $\forall f \in L^1(G) \cap L^2(G)$,*

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \mathrm{Tr}\,(\zeta(f)\zeta(f)^*)\, d\mu(\zeta). \tag{1.7}$$

The support of μ is the reduced dual $\widehat{G}^r$. □

We call μ the Plancherel measure. Note that it is implicit in the statement of the theorem that the operators $\zeta(f)\zeta(f)^*$ are of trace class. Here $\zeta(f)$ alludes, of course, to the corresponding representation of the group C^*-algebra.

The above theorem can be cast in the form of a direct integral decomposition of the regular representation:

$$L^2(G) \cong \int_{\widehat{G}} \mathcal{H}_\lambda \hat{\otimes} \mathcal{H}_\lambda^*\, d\mu(\lambda) \cong \int_{\widehat{G}} HS(\mathcal{H}_\lambda)\, d\mu(\lambda),$$

where L is the left regular representation of G on $L^2(G)$, each $\mathcal{H}_\lambda$ is a representation space of λ, $\hat{\otimes}$ denotes the completed Hilbert space tensor product,

[18] See §A.2.

[19] See Cowling, Haagerup, and Howe [17]. A *diagonal* matrix coefficient is of the form $x \mapsto \langle x \cdot u, u\rangle$. Note that the converse is false for non-compact abelian groups.

and $HS(\mathcal{H}_\lambda)$ is the space of Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. In these notes we will not concern ourselves with the explicit formula for the Plancherel measure, knowledge of its support being sufficient for our aims. We note however that Harish-Chandra has given the solution for real semisimple groups [53, 54, 55].

Definition 1.5.9 A unitary irreducible representation is *integrable* if it has an L^1 matrix coefficient.

Since matrix coefficients are bounded, integrable representations are automatically square integrable. These restrictions on the growth of matrix coefficients have consequences for the Fell topology:

Theorem 1.5.10 *Let $\pi \in \widehat{G}$ and G_0 denote the identity component of G.*

1. If G/G_0 is compact and π is square integrable, then $\{\pi\}$ is open in $\widehat{G}^r$ (Green [43]).

2. If π is integrable, then $\{\pi\}$ is open in $\widehat{G}$ (Duflo and Moore [24]). □

Recall that in the compact case the dimension gives the multiplicity of the irreducible representation in $L^2(G)$. That property is also possessed by the formal degree in the general case:

Theorem 1.5.11 *In the above situation, let $\pi \in \widehat{G}$. Then π is square integrable if and only if $\mu(\{\pi\}) > 0$, and then $\mu(\{\pi\}) = d_\pi$.*

Proof. [21, p369] □

If π is integrable and f is an L^1 matrix coefficient of π then the last result, together with Theorem 1.5.4, implies that the integrand on the right hand side of (1.7) is supported on $\{\pi\}$.

We complete our comparisons with the compact case by considering the notion of character in the non-compact case. Since an irreducible unitary representation π is now usually infinite dimensional, the operators $\pi(g)$, $g \in G$, will generally not be of trace class. However these are approximated by the $\pi(f)$, $f \in L^1(G)$, and $\pi(f)$ may be of trace class.

Therefore, let D_π be a subspace of $L^1(G)$ such that for every $f \in D_\pi$, $\pi(f)$ is of trace class. The *global character* of π is then defined to be the functional Θ_π defined as follows:

$$\Theta_\pi(f) = \operatorname{Tr} \pi(f), \quad f \in D_\pi.$$

Naturally, the usefulness of this notion, in any particular instance, will depend on the size of D_π. Ideally, it should be a dense self-adjoint subalgebra of $L^1(G)$, independent of π, and with a topology in which each Θ_π is continuous.

Suppose we are in the above situation. Consider the action of G on $L^1(G)$ by conjugation: $(x \cdot f)(y) = f(x^{-1}yx)$. Then it is trivial to see that $\pi(x \cdot f) = \pi(x)\pi(f)\pi(x)^{-1}$. Hence if $\pi(f)$ is a trace class operator, then so is $\pi(x \cdot f)$ and it has the same trace. For this reason $\Theta_\pi(f)$ is called an *invariant distribution.*

1.6 Mackey Induction

In this section, we consider an inductive construction of representations due to Mackey [98, 99]. It generalizes a method first developed for finite groups. We quickly recall that procedure. Let G be a finite group, H a subgroup of G, and (η, V) a finite dimensional representation of H. We will use η to construct a representation of G.

Consider the quotient space $G \times_H V = (G \times V)/\sim$, where the equivalence relation is defined by $(gh, v) \sim (g, \eta(h)v)$ for any $g \in G$, $h \in H$, and $v \in V$. The space $G \times_H V$ forms a vector bundle over the finite set G/H under the projection $[(g, v)] \mapsto [g]$, where the square brackets denote equivalence classes. The *induced representation* has the representation space

$$\mathrm{Ind}_H^G(\eta) = \Gamma(G \times_H V) \tag{1.8}$$

consisting of sections of the bundle $G \times_H V$, and the action of G is by left translation. Another picture for the induced representation is to consider the function space

$$C(G/H, \eta) = \left\{ f : G \to V \middle| f(gh) = \eta(h)^{-1} f(g) \ \forall h \in H, g \in G \right\}, \tag{1.9}$$

and the action of G is again by left translation: $(x \cdot f)(g) = f(x^{-1}g)$. These two realizations of the induced representation are connected as follows. Suppose $f \in C(G/H, \eta)$. Define $\sigma_f \in \Gamma(G \times_H V)$ by $\sigma_f[g] = (g, f(g))$. Conversely, suppose $\sigma \in \Gamma(G \times_H V)$. Number the cosets of G/H by an index i, and pick a representative g_i from the i-th coset. Define $f_\sigma(g_i)$ by $\sigma[g_i] = (g_i, f_\sigma(g_i))$, and for $g = g_i h$ $(h \in H)$ define $f_\sigma(g) = \eta(h)^{-1} f(g_i)$. The maps $\sigma \mapsto f_\sigma$ and $f \mapsto \sigma_f$ provide an equivalence between the left translation actions of G on $\Gamma(G \times_H V)$ and $C(G/H, \eta)$.

Now let G be a locally compact group and let H be a closed subgroup of G. We want to take a unitary representation of H and induce a unitary representation of G from it. If G/H has a Borel measure which is G-invariant under left translation, then we can consider the L^2 sections of the induced bundle (which is defined as above) and the left action of G on these sections will be unitary.

Theorem 1.6.1 *If H is a closed subgroup of a locally compact group G and $\triangle_G\big|_H = \triangle_H$, then G/H has a (left) G-invariant Borel measure which is unique upto constants.* $\square$

Thus, such an invariant measure exists if both G and H are unimodular. In general, there is a lack of such a measure, and a more roundabout approach is needed. We sketch it below.[20]

Let $\triangle_G$ and $\triangle_H$ be the modular functions for the Haar measures on G and H. Define $\triangle_{G/H} = \triangle_G / \triangle_H : H \to \mathbb{R}$. Suppose H has a continuous

[20] Our exposition is based on the appendix to Wolf [155].

representation η in a Hilbert space $\mathcal{H}_\eta$. Define:

$$C_c(G/H,\eta) = \left\{ f : G \to \mathcal{H}_\eta \;\middle|\; \begin{array}{l} 1.\ f \text{ is continuous} \\ 2.\ f \text{ is compactly supported mod } H \\ 3.\ f(gh) = \eta(h)^{-1}f(g)\ \forall g \in G,\ h \in H \end{array} \right\}.$$

Define $\tau : C_c(G) \to C_c(G/H, \triangle_{G/H}^{-1})$ by $\tau(f)(g) = \int_H f(gh)\triangle_{G/H}^{-1}(h)\,dh$. Then (we assert without proof) τ is a surjective map, and if $\tau(f) \equiv 0$ then $\int_G f(g)\,dg = 0$. So integration on G induces a linear functional on $C_c(G/H, \triangle_{G/H}^{-1})$ which we write as an integral:

$$\int_{G/H} F(x)\,dx = \int_G f(g)\,dg \qquad \text{for } F = \tau(f).$$

This is a continuous positive G-invariant integral on G/H. Now consider the representation $\eta \otimes \triangle_{G/H}^{-1/2}$, where we take η to be a unitary representation of H. Let $F_1, F_2 \in C_c(G/H, \eta \otimes \triangle_{G/H}^{-1/2})$. Then

$$\begin{aligned} \langle F_1(gh), F_2(gh)\rangle &= \langle \triangle_{G/H}^{1/2}(h)\eta(h)^{-1}F_1(g), \triangle_{G/H}^{1/2}(h)\eta(h)^{-1}F_2(g)\rangle \\ &= \triangle_{G/H}(h)\langle F_1(g), F_2(g)\rangle \qquad \forall g \in G,\ h \in H. \end{aligned}$$

Therefore the function $g \mapsto \langle F_1(g), F_2(g)\rangle$ is in $C_c(G/H, \triangle_{G/H}^{-1})$ and we can define an inner product on $C_c(G/H, \eta \otimes \triangle_{G/H}^{-1/2})$ by

$$\langle F_1, F_2\rangle = \int_{G/H} \langle F_1(x), F_2(x)\rangle\,dx. \tag{1.10}$$

Let us denote the Hilbert space completion of $C_c(G/H, \eta\otimes\triangle_{G/H}^{-1/2})$ by $\mathrm{Ind}(\eta) = \mathrm{Ind}_H^G(\eta)$. Then G acts unitarily on it by left translations:

$$(g \cdot F)(y) = F(g^{-1}y).$$

If the action of H on $\mathcal{H}_\eta$ is non-unitary, we can still get a G action by left translation on $C_c(G/H, \eta \otimes \triangle_{G/H}^{-1/2})$, but we cannot put the inner product (1.10) on it and complete to get a unitary G action on a Hilbert space. However, it may be possible to put a *different* inner product on it and thus still get a unitary representation of G. We shall do this explicitly in Chapter 3 to obtain the complementary series of $SL(2,\mathbb{R})$.

Geometrically, we may think of our representation space as the L^2 sections of the induced bundle $G \times_H (\eta \otimes \triangle_{G/H}^{-1/2}) \to G/H$. The following theorem contains some basic properties of induced representations.

Theorem 1.6.2 *Let G be a locally compact group and H a closed subgroup. Let η be a unitary representation of H. Then*

1. $\mathrm{Ind}_H^G(\eta^c) \cong \mathrm{Ind}_H^G(\eta)^c$.
2. *If there is a direct integral decomposition $\eta \cong \int_Z \eta_z\,d\mu(z)$ then*

$$\mathrm{Ind}_H^G(\eta) \cong \int_Z \mathrm{Ind}_H^G(\eta_z)\,d\mu(z).$$

3. *(Double induction) If H' is a closed subgroup of G containing H then*

$$\mathrm{Ind}_H^G(\eta) \cong \mathrm{Ind}_{H'}^G(\mathrm{Ind}_H^{H'}(\eta)).$$

□

Example 1.6.3 Let $H = \{e\}$ be the trivial subgroup of a locally compact group G, and $\eta = 1$ the trivial one dimensional representation of H. Then it is trivial to check that $C_c(G/H, \eta \otimes \triangle_{G/H}^{-1/2}) = C_c(G)$, and the inner product (1.10) is the standard L^2 inner product. Hence $\mathrm{Ind}_H^G(1) = L$, the left regular representation of G. □

A useful phenomenon for induced representations of finite groups is the multiplicity formula known as *Frobenius reciprocity*. This generalizes directly when G is compact.[21]

Theorem 1.6.4 *Let G be a compact group and H a closed subgroup. Let V and W be finite dimensional irreducible unitary representations of G and H respectively. Then we have a natural isomorphism:*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(V, W).$$

Proof. The isomorphism $\Phi : \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \to \mathrm{Hom}_H(V, W)$ is defined by $(\Phi T)(v) = (Tv)(1)$. Its inverse map Ψ is defined by $\Psi S(v)(g) = S(g^{-1}v)$. The only non-obvious part is that Ψ has the right range. □

By the multiplicity of W in V, denoted $m(W : V)$, we will mean the number of independent copies of W in V (as H representations).

Corollary 1.6.5 *In the situation of the previous theorem, we have*

$$m(V : \mathrm{Ind}_H^G(W)) = m(W : V).$$

Proof. Schur's lemma implies that $m(W : V) = \dim_{\mathbb{C}} \mathrm{Hom}_H(V, W)$. □

[21] Frobenius reciprocity exists also in more general situations. For example, Mautner [103] gives a version when G is arbitrary locally compact and H is compact.