Appendix A

A.0 Borel Measures

Let X be a set. A *Borel structure* on X is a choice of a σ-algebra $\mathcal{B}$: A collection of subsets of X which contains $\emptyset$ and is closed under countable union and complement. The pair $(X, \mathcal{B})$ is then called a *Borel space*. The members of $\mathcal{B}$ are called *Borel sets*.

If $\mathcal{A}$ is any collection of subsets of X then there is a smallest σ-algebra containing $\mathcal{A}$, and it is said to be the one *generated* by $\mathcal{A}$.

If $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are Borel spaces, then a map $f : X \to Y$ is called a *Borel map* if $E \in \mathcal{C}$ implies $f^{-1}(E) \in \mathcal{B}$.

If X is a topological space, then there is a natural associated Borel structure: The one generated by the open subsets of X. This Borel structure is said to be subordinate to the topology.

A *Borel measure* on a Borel space $(X, \mathcal{B})$ is a mapping $\mu : \mathcal{B} \to [0, \infty]$ such that

1. If $\langle E_i \rangle$ is a sequence of pair-wise disjoint members of $\mathcal{B}$ then $\mu(\cup_i X_i) = \sum_i \mu(E_i)$.
2. There is a countable subcollection $\{E_i\}$ of $\mathcal{B}$ such that $X = \cup_i E_i$ and $\mu(E_i) < \infty$ for every i.

A.1 Operator Algebras

Definition A.1.1 A *complex $*$-algebra* is a complex algebra A with a map $A \to A$, $x \mapsto x^*$, satisfying:

1. $(x + y)^* = x^* + y^*$, for every $x, y \in A$.
2. $(\lambda x)^* = \bar{\lambda} x^*$, for every $x \in A$, $\lambda \in \mathbb{C}$.
3. $(xy)^* = y^* x^*$, for every $x, y \in A$.
4. $x^{**} = x$, for every $x \in A$.

The map $x \mapsto x^*$ is called the *involution* on A. Note that we have not required our algebra to have an identity.

Definition A.1.2 A *Banach algebra* is a Banach space A which is also a complex algebra, such that $||xy|| \leq ||x||\,||y||$, for every $x, y \in A$.

Definition A.1.3 A *C^*-algebra* is a Banach algebra A which is also a $*$-algebra, such that $||xx^*|| = ||x||^2$, for every $x \in A$.[1]

If A is a C^*-algebra, then $||x^*|| = ||x||$ for every $x \in A$. If A has an identity 1, then $||1|| = 1$.

Example A.1.4 Let X be a locally compact Hausdorff space. Then the set $C_0(X)$ of complex continuous functions which vanish at ∞ is a commutative Banach algebra with respect to the sup norm and point-wise multiplication. For $f \in C_0(X)$ define $f^*(x) = \overline{f(x)}$. Then $C_0(X)$ becomes a commutative C^*-algebra.

Given a commutative C^*-algebra A, the set X_A consists of the non-zero multiplicative functionals on A. We equip X_A with the weak topology induced by the evaluation maps $\hat{x} : X_A \to \mathbb{C}$, $\hat{x}(\phi) = \phi(x)$ $(x \in A)$. Then X_A is a locally compact Hausdorff space.

Theorem A.1.5 (Gelfand-Naimark) *Every commutative C^*-algebra A is isomorphic to $C_0(X_A)$ via the Gelfand transform $x \mapsto \hat{x}$. If it has an identity, then X must be compact.* □

Example A.1.6 Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a Banach algebra with respect to the operator norm, and with composition as multiplication. The adjoint map makes it a C^*-algebra. If a subalgebra of $\mathcal{B}(\mathcal{H})$ is norm closed as well as $*$-invariant, it is itself a C^*-algebra.

Theorem A.1.7 (Gelfand-Naimark) *Every C^*-algebra is isomorphic to a norm closed and $*$-invariant subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.* □

Representations of $*$-Algebras

Consider a Banach $*$-algebra A. A *$*$-representation* of A is an algebra homomorphism $a \mapsto \pi(a)$ into the bounded operators on a Hilbert space $\mathcal{H}$ such that $\pi(a^*) = \pi(a)^*$ and $||\pi(a)|| \leq ||a||$ $\forall a \in A$. We say π is *non-degenerate* if $\pi(a)v = 0$ $\forall a \in A \implies v = 0$, while it is *irreducible* if $\mathcal{H}$ has no proper non-zero closed subspace invariant under $\pi(A)$. An irreducible π is obviously non-degenerate. Unitary equivalence of $*$-representations is defined in the usual way, and the *dual* $\widehat{A}$ of A is the set of unitary equivalence classes of irreducible $*$-representations of A.

Enveloping C^*-Algebra

Equip the Banach $*$-algebra A with the following semi-norm:

$$||a||' = \sup_{\pi} ||\pi(a)|| \quad (\leq ||a||),$$

where π varies over the $*$-representations of A. Let $I = \{a \in A : \ ||a||' = 0\}$. Then I is a self-adjoint two-sided ideal of A, and the completion $\bar{A}$ of A/I

[1] In the literature (e.g. [119]) C^*-algebras are often called B^*-algebras.

with respect to the norm induced by $||\cdot||'$ is a C^*-algebra. This completion is called the *enveloping C^*-algebra* of A. If A is itself a C^*-algebra, then this construction gives $\bar{A} = A$. The $*$-representations of A are in natural bijection with those of $\bar{A}$, as are the sub-categories of non-degenerate and irreducible representations.

Fell Topology

Let $\mathcal{V}$ be a family of $*$-representations of a Banach $*$-algebra A. We say a $*$-representation μ of A is weakly contained in $\mathcal{V}$ if

$$\bigcap_{\pi\in\mathcal{V}} \ker\pi \subseteq \ker\mu.$$

Example A.1.8 The notion of weak containment can be motivated by the following consideration. Let $\mathcal{V} = \{\pi\}$ and $\pi = \oplus\pi_i$ a direct sum. Then each π_i is weakly contained in $\mathcal{V}$. Thus weak containment tracks common subrepresentations. □

We can give $\widehat{A}$ the *Fell topology*: The closure of a subset $\mathcal{V}$ of $\widehat{A}$ consists of all the equivalence classes that are weakly contained in it.[2] The set $\widehat{A}$ is now a locally compact Baire space.

Direct Sum of C^*-Algebras

Suppose we have a family $\{\mathcal{A}_i\}_I$ of C^*-algebras, where I is an arbitrary index set. Then we define the *direct sum* $\mathcal{A} = \bigoplus_{i\in I}\mathcal{A}_i$ as follows. First, for a member $(a_i)_I$ of the direct product $\prod_{i\in I}\mathcal{A}_i$, we say $a_i \to 0$ if for every $\epsilon > 0$ there is a finite subset $F \subset I$ such that $||a_i|| < \epsilon$ whenever $i \in I \setminus F$. Now we take $\mathcal{A}$ to be the vector space of all tuples $(a_i)_I$ such that $a_i \to 0$. The vector space operations are defined component-wise. Component-wise involution makes $\mathcal{A}$ a $*$-algebra and the norm $||(a_i)_I|| = \sup_{i\in I}||a_i||$ makes it a C^*-algebra.

Von Neumann Algebras

Definition A.1.9 Let A be a $*$-algebra of bounded operators on a Hilbert space $\mathcal{H}$. Its *commutator algebra* is $A' = \{T \in \mathcal{B}(\mathcal{H}) : TX = XT\ \forall X \in A\}$. Then it is obvious that $A \subset A''$.

Definition A.1.10 A $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ is a *von Neumann algebra* if $A = A''$.

Other characterizations of von Neumann algebras are obtained from the *double commutator theorem*, which says that A'' always equals the strong and weak operator topology closures of A. In particular, A' is always a von Neumann algebra. A von Neumann algebra A is a *factor* if its center has dimension one. A is a factor if and only if A' is a factor. It is a *factor of Type I* if it is unitarily equivalent to the operator space of some Hilbert space $\mathcal{H}'$.

[2] The Fell topology is also called the *hull-kernel topology*.

A.2 Compact and Trace Class Operators

Let $\mathcal{H}$ be a separable Hilbert space. Recall (see Gelfand and Vilenkin [40] for proofs of the claims in this section) that $K \in \mathcal{B}(\mathcal{H})$ is called *compact* if it takes bounded sets to ones with compact closure. Every compact operator K has a polar decomposition $K = UT$, where T is positive-definite compact, and U is partially isometric. Fix such a $K = UT$ and let $\{\lambda_n\}$ be the eigenvalues of T. Then $||K|| = \sup_n \lambda_n$.

The compact operator K is *Hilbert-Schmidt* if there is a complete orthonormal set $\{e_i\}$ of $\mathcal{H}$ such that $\sum_i ||Ke_i||^2 < \infty$ or, equivalently, if $\sum_n \lambda_n^2 < \infty$. It is *nuclear* if $\sum_n \lambda_n < \infty$. A nuclear operator is automatically Hilbert-Schmidt.

The compact operator K is *of trace class* if for any two choices of complete orthonormal systems $\{e_i\}$ and $\{f_i\}$ of $\mathcal{H}$, $\sum_i |\langle Ke_i, f_i\rangle| < \infty$. The finite sum $\sum_i \langle Ke_i, f_i\rangle$ is then called the *trace* of K. It is independent of the choice of $\{e_i\}$ and $\{f_i\}$. A positive-definite compact operator is of trace class if and only if it is nuclear.

Nuclear operators have the following characterization:

Theorem A.2.1 *An operator is nuclear if and only if it can be expressed as the product of two Hilbert-Schmidt operators.* □

Every nuclear operator is of trace class. Another special class of operators is the following: K is *summable* if there is an orthonormal basis $\{e_i\}$ of $\mathcal{H}$ such that $\sum_{m,n} |\langle Ke_m, e_n\rangle| < \infty$. Every summable operator is of trace class.

Let $C(\mathcal{H})$ denote the compact operators on $\mathcal{H}$, $HS(\mathcal{H})$ the Hilbert-Schmidt operators, and $N(\mathcal{H})$ the nuclear operators. Each of these sets is a self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$. Consider the following quantities:

$$||K||_1 = \sum_n \lambda_n, \quad ||K||_2 = \Big[\sum_n \lambda_n^2\Big]^{1/2}.$$

The first defines a norm (the *trace norm*) on $N(\mathcal{H})$, the second a norm on $HS(\mathcal{H})$ (the *Hilbert-Schmidt norm*), such that each becomes a Hilbert space. $C(\mathcal{H})$ is itself complete relative to the operator norm (and hence is a C^*-algebra). The following inequalities hold:

$$||K|| \leq ||K||_2 \leq ||K||_1.$$

Theorem A.2.2

1. *Every non-trivial irreducible $*$-representation of $C(\mathcal{H})$ is equivalent to the identity representation $C(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H})$.*
2. *If B is a sub-C^*-algebra of $C(\mathcal{H})$ such that its identity representation is irreducible, then $B = C(\mathcal{H})$.*
3. *Let A be a C^*-algebra with an irreducible $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$, such that $\pi(A) \cap C(\mathcal{H}) \neq 0$. Then $C(\mathcal{H}) \subseteq \pi(A)$ and every irreducible $*$-representation of A whose kernel equals that of π is equivalent to π.*

Proof. See Dixmier [21, §4.1]. □

Corollary A.2.3 *Let A be any C^*-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ an irreducible $*$-representation, such that A maps into $C(\mathcal{H})$. Then π is a closed point in $\widehat{A}$ (in the hull-kernel topology).*

Proof. This is an immediate consequence of (3) of the previous theorem. □

A.3 Special Functions

Here, we write down some facts about the gamma and hyper-geometric functions that we use in Chapter 3.

Gamma Function

For $\operatorname{Re} z > 0$, we define $\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx$. This is an analytic function of z and satisfies $\Gamma(1) = 1$, as well as $\Gamma(z+1) = z\Gamma(z)$. Using this, we extend Γ to $\operatorname{Re} z > -1$, with the exception of the point 0. Repeating this process, we get the analytic continuation of Γ to the full complex plane except the points $\{0, -1, -2, \dots\}$. We have the following facts about Γ:

$$\Gamma(n+1) = n! \qquad \text{for } n \in \mathbb{Z}_{\geq 0} \tag{A.1}$$

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} \tag{A.2}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad \text{for } 0 < \operatorname{Re} z < 1 \tag{A.3}$$

(A.3) is called the *reflection formula.*

Binomial Theorem

An application of the Taylor series gives the following expansion:

$$(1-z)^\lambda = \sum_{s=0}^{\infty} (-1)^s \frac{\Gamma(\lambda+1)}{s!\,\Gamma(\lambda-s+1)} z^s \quad \text{for } |z| < 1 \tag{A.4}$$

Hypergeometric Function

The binomial theorem can be expressed also as follows:

$$(1-z)^{-\lambda} = \frac{1}{\Gamma(\lambda)} \sum_{s=0}^{\infty} \frac{\Gamma(\lambda+s)}{s!} z^s \quad \text{for } |z| < 1$$

The hypergeometric functions ${}_pF_q$ generalize the expression on the right hand side. We define, for $|z| < 1$,

$${}_pF_q(\alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; z) = \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)} \sum_{s=0}^{\infty} \frac{\Gamma(\alpha_1+s)\cdots\Gamma(\alpha_p+s)}{s!\ \Gamma(\gamma_1+s)\cdots\Gamma(\gamma_q+s)} z^s$$

Two special cases are ${}_0F_0(z) = e^z$ and ${}_1F_0(\alpha; z) = (1-z)^{-\alpha}$. The case of particular interest to us is $F = {}_2F_1$, and from now on we refer to it as *the* hypergeometric function. This also has an integral formula:

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 (1-tz)^{-\beta} t^{\alpha-1} (1-t)^{\gamma-\alpha-1} dt$$

This formula can be proved by applying the binomial theorem to the integrand, and it allows the analytic continuation of F outside $|z| < 1$. F also arises as the solution to the *hypergeometric equation*:

$$z(1-z)w'' + [\gamma - (\alpha+\beta+1)z]w' - \alpha\beta w = 0$$

Indeed, in Bargmann[6] the matrix coefficients were found by showing they satisfied a hypergeometric equation and so could be expressed in terms of the hypergeometric function. That differential equation was obtained by an application of the Casimir operator.

Bessel Functions

Consider the following formal expansion:

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z) t^n.$$

If we let z be a complex variable then each J_n becomes a holomorphic function and is called the *Bessel function of order n*. In particular, we have the following expression for the Bessel function of order zero:

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{z}{2}\right)^{2m}.$$

The function J_0 has the following integral expression,

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\mathrm{Re}\, z e^{i\theta}}\, d\theta, \tag{A.5}$$

which can be proven by expanding the integrand on the right side in a power series and expanding term by term. In Chapter 4, we utilize "Weber's infinite integral formula" [149, p391], given below:

$$\int_0^{\infty} t^{\lambda-1} J_0(t)\, dt = 2^{\lambda-1} \frac{\Gamma(\lambda/2)}{\Gamma(1-\lambda/2)}, \qquad 0 < \lambda < \frac{1}{2}. \tag{A.6}$$

A.4 Homological Algebra

In this section we isolate some of the definitions and properties regarding injective and projective modules. There are many undefined terms, for instance: category, object, morphism, functor, covariant, contravariant, left exact, right exact, Hom, $\otimes$, complex, exact sequence, natural map. These the reader should look up in some standard reference such as Jacobson [70], or in the appendices to Knapp and Vogan [81].

Definition A.4.1 Let R be a ring. A *good category over R* has as objects some class of left R modules which is closed under passage to submodules, quotients, and finite direct sums; and as morphisms all the R linear maps between its objects. If R is an associative algebra over a field k we also require the morphisms to be k linear.

Let $\mathcal{C}$ be a good category over a ring R. If A and B are objects in $\mathcal{C}$ the set of morphisms between them is denoted by $\mathrm{Hom}_R(A, B)$.

Definition A.4.2 A module P in $\mathcal{C}$ is called *projective* if given modules $B, C \in \mathcal{C}$, a surjective morphism $\psi : C \to B$, and a morphism $\tau : P \to B$, there is a map $\sigma : P \to C$ such that $\psi \circ \sigma = \tau$.

$$\begin{tikzcd} & P \arrow[d, "\tau"'] \arrow[dr, "\sigma"] & \\ 0 & B \arrow[l] & C \arrow[l, "\psi"'] \end{tikzcd}$$

Definition A.4.3 A module I in $\mathcal{C}$ is called *injective* if given modules $B, C \in \mathcal{C}$, an injective morphism $\psi : B \to C$, and a morphism $\tau : B \to I$, there is a map $\sigma : C \to I$ such that $\sigma \circ \psi = \tau$.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & B \arrow[u, "\tau"] \arrow[r, "\psi"] & C \arrow[ul, "\sigma"'] \end{tikzcd}$$

Definition A.4.4 An exact sequence in $\mathcal{C}$,

$$0 \longrightarrow A \xrightarrow{\sigma} B \xrightarrow{\tau} C \longrightarrow 0,$$

is *split* if either of the following equivalent conditions holds:

1. $\exists\ \bar{\sigma} \in \mathrm{Hom}_R(B, A)$ such that $\bar{\sigma} \circ \sigma = 1_A$.
2. $\exists\ \bar{\tau} \in \mathrm{Hom}_R(C, B)$ such that $\tau \circ \bar{\tau} = 1_C$.

In the above situation, if A is injective, or C is projective, then the sequence must be split.

Definition A.4.5 Suppose $\mathcal{C}$ and $\mathcal{C}'$ are good categories over rings R and R' respectively. A functor $F : \mathcal{C} \to \mathcal{C}'$ is *additive* if

$$F(\phi_1 + \phi_2) = F(\phi_1) + F(\phi_2)$$

whenever $\phi_1, \phi_2 \in \mathrm{Hom}_R(A, B)$ for any $A, B \in \mathcal{C}$.

From here onwards, all functors will be additive.

Theorem A.4.6 *Fix a good category $\mathcal{C}$ over a ring R and let $\mathcal{C}'$ be the good category of $\mathbb{Z}$ modules. Fix a module V in $\mathcal{C}$. Then*

1. $\mathrm{Hom}_R(\,\cdot\,, V) : \mathcal{C} \to \mathcal{C}'$ *is a contravariant left exact functor.*
2. $\mathrm{Hom}_R(V, \,\cdot\,) : \mathcal{C} \to \mathcal{C}'$ *is a covariant left exact functor.*
3. *Let U be a right R module. Then $U \otimes_R (\cdot) : \mathcal{C} \to \mathcal{C}'$ is a covariant right exact functor.* $\square$

When R is an associative algebra over k, $\mathcal{C}'$ in the above theorem can be taken to be the category of vector spaces over k.

Theorem A.4.7 *Let $\mathcal{C}$ be a good category over R.*

1. $P \in \mathcal{C}$ is a projective module if and only if the functor $\mathrm{Hom}_R(\,\cdot\,, P)$ *is exact.*

2. $I \in \mathcal{C}$ is an injective module if and only if the functor $\mathrm{Hom}_R(I, \,\cdot\,)$ *is exact.* □

Definition A.4.8 Given $M \in \mathcal{C}$, a *projective resolution* of M is an exact sequence in $\mathcal{C}$,

$$0 \longleftarrow M \longleftarrow X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots,$$

where each X_i is a projective module. An *injective resolution* of M is an exact sequence in $\mathcal{C}$,

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots,$$

where each X_i is an injective module.

Definition A.4.9 We say $\mathcal{C}$ has *enough projectives (injectives)* if every module in $\mathcal{C}$ has a projective (injective) resolution.

We assume from now on that our categories have enough projectives and injectives. Let $\mathcal{C}$ and C' be good categories over R and R' respectively, and $F : \mathcal{C} \to \mathcal{C}'$ a covariant left exact functor. Let M be a module in $\mathcal{C}$, with an injective resolution

$$0 \longrightarrow M \xrightarrow{\varepsilon} X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \longrightarrow \cdots.$$

Apply F to each term in this resolution, and drop the $F(M)$ term to get the complex

$$0 \longrightarrow F(X_0) \xrightarrow{F(\phi_0)} F(X_1) \xrightarrow{F(\phi_1)} \cdots.$$

The *right derived functor* $F^n : \mathcal{C} \to \mathcal{C}'$ takes the module M to the module $F(X_{m+1})/F(X_m)$. To see F^n at the level of morphisms, let $\psi \in \mathrm{Hom}_R(M, M')$, where M' is another object in $\mathcal{C}$. Let X'_* be an injective resolution of M'. Then ψ induces maps $\psi_n : X_n \to X'_n$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\psi"'] & X_0 \arrow[r] \arrow[d, "\psi_0"'] & X_1 \arrow[r] \arrow[d, "\psi_1"'] & \cdots \\
0 \arrow[r] & M' \arrow[r] & X'_0 \arrow[r] & X'_1 \arrow[r] & \cdots
\end{tikzcd}$$

The maps ψ_n induce maps at the cohomology level, and these are called $F^n(\psi)$. The functor F^0 is naturally isomorphic to F.

Similarly, if $F : \mathcal{C} \to \mathcal{C}'$ is a covariant right exact functor, we use projective resolutions to define its *left derived functors* F_n. The constructions of derived functors are independent of the particular choice of resolutions.

Theorem A.4.10 *Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be good categories over R, R' and R'' respectively. Suppose $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ are covariant functors, and F is exact.*

1. *If G is left exact, then $G \circ F$ is left exact and $(G \circ F)^n \cong G^n \circ F$.*
2. *If G is right exact, then $G \circ F$ is right exact and $(G \circ F)_n \cong G_n \circ F$.*

$\square$

Now we give some results that help in identifying naturally isomorphic functors. Suppose R and S are algebras over a field k, with approximate identities. We use the notation ${}^{R}A$ to indicate a left R module A, and the obvious modifications for right modules and S modules. First, suppose we have A^R, ${}^{R}B^S$ and C^S. Then there is a k vector-space isomorphism

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \tag{A.7}$$

which is natural in A, B and C. The isomorphism Φ from left to right is given by

$$(\Phi(\phi)(a))(b) = \phi(a \otimes b). \tag{A.8}$$

Similarly, suppose ${}^{R}A$, ${}^{S}B^R$ and ${}^{S}C$. Then there is a natural k vector-space isomorphism

$$\mathrm{Hom}_S(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \tag{A.9}$$

Finally, suppose A^R, ${}^{R}B^S$ and ${}^{S}C$. Then there is a natural k vector-space isomorphism

$$(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C.) \tag{A.10}$$

Theorem A.4.11 *Let $\mathcal{C}$ and $\mathcal{C}'$ be good categories of modules over R and S respectively. Suppose $F, G : \mathcal{C} \to \mathcal{C}'$ are functors s.t.*

$$\mathrm{Hom}_S(U, F(A)) \cong \mathrm{Hom}_S(U, G(A))$$

naturally in $A \in \mathcal{C}$ and $U \in \mathcal{C}'$. Then $F(A) \cong G(A)$ naturally in A. $\square$

In the above theorem, the isomorphism $F(A) \to G(A)$ is given by the element of $\mathrm{Hom}_S(F(A), G(A))$ which corresponds to $1 \in \mathrm{Hom}_S(F(A), F(A))$. The conclusion also holds if we are given, instead, that $\mathrm{Hom}_S(F(A), V) \cong \mathrm{Hom}_S(G(A), V)$.

A.5 Lie Algebra Cohomology

Let M be a manifold, $\mathfrak{X}(M)$ the space of its C^∞ vector fields, and $\Lambda^k(M)$ its C^∞ k-th differential forms. The deRham cohomology of M (See, for instance, chapters V and VI of [12]) is given by the differential map $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ defined by

$$\begin{aligned}(d\varphi)(X_1, \ldots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i \cdot (\varphi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),\end{aligned}$$

where $\varphi \in \Lambda^k(M)$ and $X_1, \ldots X_{k+1} \in \mathfrak{X}(M)$. When $M = G$, a Lie group, the differential d restricts to the left invariant forms and gives a cohomology of the Lie algebra of G. When G is compact this Lie algebra cohomology is isomorphic to the de Rham cohomology of G.

More generally, let $\mathfrak{g}$ be a complex Lie algebra and V a $\mathcal{U}(\mathfrak{g})$ module. The space of *k-th cochains with coefficients in V* is defined to be

$$C^k = C^k(\mathfrak{g}, V) = \Lambda^k(\mathfrak{g}^*) \otimes V,$$

the skew-symmetric multi-linear functions in k variables from $\mathfrak{g}$ and taking values in V. C^k is itself a $\mathcal{U}(\mathfrak{g})$ module under the action:

$$\begin{aligned}(X \cdot c)(X_1, \ldots, X_k) &= \sum_j (-1)^j c(X_1, \ldots, [X, X_j], \ldots, X_k) \\ &\quad + X \cdot (c(X_1, \ldots, X_k)),\end{aligned}$$

where $c \in C^i$ and $X, X_1, \ldots, X_i \in \mathfrak{g}$. Now define a differential map $d : C^k \to C^{k+1}$ by

$$\begin{aligned}(dc)(X_1, \ldots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i \cdot (c(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}).\end{aligned}$$

This gives us a complex:

$$\{0\} \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \cdots \xrightarrow{d} C^N \longrightarrow 0,$$

where $N = \dim \mathfrak{g}$ and the differential d is a homomorphism of $\mathcal{U}(\mathfrak{g})$ modules. Define $Z^k(\mathfrak{g}, V) = \ker d \cap C^k(\mathfrak{g}, V)$ and $B^k(\mathfrak{g}, V) = \operatorname{im} d \cap C^k(\mathfrak{g}, V)$, and the cohomology group

$$H^k(\mathfrak{g}, V) = \frac{Z^k(\mathfrak{g}, V)}{B^k(\mathfrak{g}, V)}.$$

Then $\mathfrak{g}$ has the trivial action on $H^k(\mathfrak{g}, V)$. Further, consider the special case $H^0(\mathfrak{g}, V)$. It is easy to see that

$$H^0(\mathfrak{g}, V) = Z^0(\mathfrak{g}, V) \cong \{v \in V : \ Xv = 0 \ \forall X \in \mathfrak{g}\},$$

where we use the identification $C^0(\mathfrak{g}, V) \cong V$. Thus $H^0(\mathfrak{g}, \cdot)$ is naturally isomorphic to the *invariants functor* $V \to V^{\mathfrak{g}}$ from the good category $\mathcal{C}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ modules to the category of complex vector spaces.

The above constructions can be rewritten in the language of derived functors. Let $X_n = \mathcal{U}(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g} \in \mathcal{C}(\mathfrak{g})$, where $\mathfrak{g}$ acts trivially on $\Lambda^n \mathfrak{g}$. The *Koszul resolution* is the exact sequence

$$0 \longleftarrow \mathbb{C} \xleftarrow{\varepsilon} X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} X_2 \longleftarrow \cdots,$$

where $\varepsilon : X_0 \cong \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ is the projection to the constant term, and $\partial : X_n \to X_{n-1}$ is given by

$$\begin{aligned}\partial(u \otimes X_1 \wedge \cdots \wedge X_n) &= \sum_i (-1)^{i+1} u X_i \otimes (X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_n)) \\ &\quad + \sum_{i<j} (-1)^{i+j} (u \otimes [X_i, X_j] \wedge X_1 \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n).\end{aligned}$$

This gives a projective resolution of the trivial representation $\mathbb{C}$. Applying the $\mathrm{Hom}_{\mathbb{C}}(\,\cdot\,, V)$ functor to this resolution, where $V \in \mathcal{C}(\mathfrak{g})$, we get the following complex:

$$0 \longrightarrow V \longrightarrow \mathrm{Hom}_{\mathbb{C}}(X_0, V) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(X_1, V) \longrightarrow \cdots \tag{A.11}$$

Theorem A.5.1 (Koszul [87]) *The above complex is an injective resolution of V.* $\square$

Applying the invariants functor to (A.5) we get back the complex $0 \to V \to C^*$, for

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(X_n, V)^{\mathfrak{g}} &= \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}) \otimes \Lambda^n \mathfrak{g}, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), V)) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, V) \;\cong\; \Lambda^n(\mathfrak{g}^*) \otimes V, \end{aligned}$$

and hence the $H^*(\mathfrak{g}, \,\cdot\,)$ are the right derived functors of the invariants functor.