Workshop on Group Theory: Classification of Reductive Algebraic Groups
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Introduction to Lie Algebras

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Lecture 5
Jordan Decomposition, Cartan's Criterion and the Killing Form

Jordan Decomposition

We keep in place the requirement that $\mathbb{F}$ is algebraically closed, but allow arbitrary characteristic.

Diagonalizable linear maps over an algebraically closed field are also called *semisimple*.

We have the following standard result from linear algebra:

Theorem 1 (Jordan Decomposition) *Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. Then any $X \in L(V)$ has a unique decomposition $X = S + N$ where S is semisimple, N is nilpotent and $[S, N] = 0$. Moreover,*

1. *There exist polynomials p, q without constant term such that $p(X) = S$ and $q(X) = N$.*
2. *For any eigenvalue λ of X, define its* generalized eigenspace *by*

$$V_\lambda = \{v \in V : (X - \lambda I)^k v = 0 \text{ for some } k\}.$$

Then V is the direct sum of the V_λ's and S acts on V_λ by λ.

S and N are called (respectively) the semisimple and nilpotent parts of X. ■

Exercise 2 *Let $X = S+N$ be the Jordan decomposition of $X \in L(V)$. Then*

1. *If $M \in L(V)$ commutes with X, it commutes with S and N.*
2. *If $A \subset B \subset V$ are subspaces such that $X(B) \subset A$, then $S(B), N(B) \subset A$.*

Exercise 3 *Let $X \in \mathfrak{g} = gl(V)$ have Jordan decomposition $X = S + N$. Then* $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ *has Jordan decomposition*

$$\mathrm{ad}(X) = \mathrm{ad}(S) + \mathrm{ad}(N).$$

Theorem 4 *Let $\mathfrak{g}$ be a Lie algebra. Then* $\mathrm{Der}(\mathfrak{g})$ *contains the semisimple and nilpotent parts of all its elements.*

Proof. Let $D \in \mathrm{Der}(\mathfrak{g})$ with Jordan decomposition $D = S + N$. We have to show $S \in \mathrm{Der}(\mathfrak{g})$. Consider the generalized eigenspaces of D:

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : (D - \lambda I)^k X = 0 \text{ for some } k\}.$$

To see that S is a derivation, it is enough to apply it to brackets of the form $[X, Y]$ where $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$. We first show that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Note that

$$\begin{aligned}(D - \lambda - \mu)[X, Y] &= [DX, Y] + [Y, DX] - \lambda[X, Y] - \mu[X, Y] \\ &= [(D - \lambda)X, Y] + [X, (D - \mu)Y].\end{aligned}$$

This is easily generalized by induction to

$$(D - \lambda - \mu)^n [X, Y] = \sum_{i=0}^{n} {}^nC_i \, [(D - \lambda)^i X, (D - \mu)^{n-i} Y].$$

Let $(D - \lambda I)^k X = (D - \mu I)^k Y = 0$. Then it follows that

$$(D - (\lambda + \mu)I)^{2k} [X, Y] = 0.$$

Therefore, for $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$, we have

$$S[X, Y] = (\lambda + \mu)[X, Y] = [\lambda X, Y] + [X, \mu Y] = [SX, Y] + [X, SY].$$

The direct sum decomposition $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$ now implies that $S \in \mathrm{Der}(\mathfrak{g})$. $\square$

Cartan's Criterion

We assume that $\mathbb{F}$ is algebraically closed and that $\operatorname{char}(\mathbb{F}) = 0$.

For a linear map $X \in L(V)$ the following criterion for solvability is quite easy to obtain: Let $A \subset B \subset V$ be subspaces, and define

$$M = \{T \in L(V) : T(B) \subset A\}.$$

Suppose $X \in M$ satisfies $\operatorname{Tr}(XT) = 0 \ \forall T \in M$. Then X is nilpotent.

We have a version of this for the adjoint action.

Lemma 5 *Let $A \subset B$ be subspaces of $L(V)$. Define*

$$M = \{X \in \mathfrak{gl}(V) : [X, B] \subset A\}.$$

Suppose $X \in M$ satisfies $\operatorname{Tr}(XY) = 0$ for all $Y \in M$. Then X is nilpotent.

Proof. Let X have Jordan decomposition $S + N$. Fix a basis of V in which S has a diagonal matrix:

$$[S] = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}.$$

Since $\operatorname{char}(\mathbb{F}) = 0$ we have the rationals $\mathbb{Q} \subset \mathbb{F}$. Treat $\mathbb{F}$ as a vector space over $\mathbb{Q}$ and let E be the span (over $\mathbb{Q}$) of $a_1, \ldots, a_n$. We have to show $E = 0$ (as that will give $S = 0$).

We will show $E^* = 0$. Let $f : E \to \mathbb{Q}$ be linear. Define $Y \in \mathfrak{gl}(V)$ by

$$[Y] = \begin{pmatrix} f(a_1) & & \\ & \ddots & \\ & & f(a_n) \end{pmatrix}.$$

Then $\operatorname{ad}(S)$ has eigenvalues $a_i - a_j$ and $\operatorname{ad}(Y)$ has eigenvalues $f(a_i) - f(a_j)$, in both cases corresponding to the eigenvectors E_{ij}. By Lagrange interpolation there is a polynomial $r(t) \in \mathbb{F}[t]$ such that

$$r(a_i - a_j) = f(a_i) - f(a_j), \quad \forall i, j.$$

It follows that $r(\operatorname{ad}(S)) = \operatorname{ad}(Y)$. Note that r has zero constant term.

We know $\mathrm{ad}(S)$ is a polynomial in $\mathrm{ad}(X)$ without constant term, hence $\mathrm{ad}(Y)$ is itself a polynomial in $\mathrm{ad}(X)$ without constant term.

Since $\mathrm{ad}(X)$ maps B into A, so does $\mathrm{ad}(Y)$. Hence $\mathrm{ad}(Y) \in M$. Therefore $\mathrm{Tr}\,(XY) = 0$, which gives $\sum_i a_i f(a_i) = 0$. Applying f, we get $\sum_i f(a_i)^2 = 0$, and hence $f(a_i) = 0\ \forall i$. $\square$

Exercise 6 *If $X, Y, Z \in L(V)$ then* $\mathrm{Tr}\,([X,Y]Z) = \mathrm{Tr}\,(X[Y,Z])$.

Recall the earlier exercise that $\mathfrak{g}$ solvable implies $\mathrm{Tr}\,(\mathrm{ad}(X)\mathrm{ad}(Y)) = 0$ for every $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. We shall establish the converse.

Theorem 7 (Cartan's Criterion for Solvability) *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra such that* $\mathrm{Tr}\,(XY) = 0$ *for every $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

Proof. Choose $A = [\mathfrak{g},\mathfrak{g}]$, $B = \mathfrak{g}$, and define

$$M = \{T \in \mathfrak{gl}(V) : [T,B] \subset A\}.$$

Clearly, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \subset M$. Now let $X, Y \in \mathfrak{g}$ and $T \in M$. Then

$$\mathrm{Tr}\,([X,Y]T) = \mathrm{Tr}\,(X[Y,T]) = \mathrm{Tr}\,([Y,T]X).$$

By definition of M, $[T,Y] \in B$, hence by the hypothesis of the theorem $\mathrm{Tr}\,([Y,T]X) = 0$. So we have obtained that $\mathrm{Tr}\,([X,Y]T) = 0$ for every $T \in M$. It follows that $\mathrm{Tr}\,(ZT) = 0$ for every $Z \in [\mathfrak{g},\mathfrak{g}]$ and $T \in M$. The earlier Lemma therefore implies that Z is nilpotent for each $Z \in [\mathfrak{g},\mathfrak{g}]$.

Therefore $[\mathfrak{g},\mathfrak{g}]$ is nilpotent and $\mathfrak{g}$ is solvable. $\square$

Exercise 8 *Let $\mathfrak{g}$ be a Lie algebra such that* $\mathrm{Tr}\,(\mathrm{ad}(X)\mathrm{ad}(Y)) = 0$ *for every $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

Killing Form

Definition 9 Let $\mathfrak{g}$ be a Lie algebra. Its *Killing form* $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is defined by $\kappa(X,Y) = \mathrm{Tr}\,(\mathrm{ad}(X)\mathrm{ad}(Y)$.

Exercise 10 *The Killing form of $\mathfrak{g}$ is bilinear, symmetric and associative: $\kappa([X,Y],Z) = \kappa(X,[Y,Z])$.*

Definition 11 Given a bilinear symmetric form $\beta : V \times V \to \mathbb{F}$, its *radical* is defined to be

$$S = \{x \in V : \beta(x, y) = 0 \ \forall y \in V\}.$$

If $S = 0$ we call the form non-degenerate.

Exercise 12 *The radical of the Killing form of* $\mathfrak{g}$ *is an ideal of* $\mathfrak{g}$.

Exercise 13 *Let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$. *Let* κ *be the Killing form of* $\mathfrak{g}$ *and* κ_h *the Killing form of* $\mathfrak{h}$. *Then* $\kappa_h = \kappa|_{\mathfrak{h}\times\mathfrak{h}}$.

For the remaining results we again restrict to a field $\mathbb{F}$ which is algebraically closed with $\mathrm{char}(\mathbb{F}) = 0$.

First, note that the Cartan Criterion corollary and its converse can now be expressed as: $\mathfrak{g}$ is solvable iff the radical of its Killing form contains $[\mathfrak{g}, \mathfrak{g}]$.

Exercise 14 *The radical of the Killing form of* $\mathfrak{g}$ *is solvable. (Hence the radical of the Killing Form is in* $\mathrm{Rad}(\mathfrak{g})$*.)*

Theorem 15 (Criterion for Semisimplicity) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Killing form* κ *is non-degenerate.*

Proof. Let S be the radical of the Killing form κ of $\mathfrak{g}$.

If $\mathfrak{g}$ is semisimple, $S \subset \mathrm{Rad}(\mathfrak{g}) = 0$.

Now, suppose $\mathrm{Rad}(\mathfrak{g}) \neq 0$. Then the last non-zero term $\mathfrak{h}$ in its derived series is an abelian ideal of $\mathfrak{g}$. If $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ then $\mathrm{ad}(X)\mathrm{ad}(Y)$ maps $\mathfrak{g} \to \mathfrak{h}$ and $(\mathrm{ad}(X)\mathrm{ad}(Y))^2$ maps $\mathfrak{g} \to [\mathfrak{h}, \mathfrak{h}] = 0$. Therefore $\mathrm{ad}(X)\mathrm{ad}(Y)$ is nilpotent and so $\kappa(X, Y) = 0$. This shows $\mathfrak{h} \subset S \neq 0$. □

Remark The first half of the proof uses Cartan's Criterion and so needs $\mathbb{F}$ algebraically closed and $\mathrm{char}(\mathbb{F}) = 0$. The second half works for any $\mathbb{F}$ and shows that for any $\mathfrak{g}$ every abelian ideal is in S.

Exercise 16 *Let* $\mathfrak{g} = \mathbb{F}$. *Give representations of* $\mathfrak{g}$ *in which:*

1. *Every element of* $\mathfrak{g}$ *acts semisimply.*
2. *Every element of* $\mathfrak{g}$ *acts nilpotently.*
3. *No element of* $\mathfrak{g}$ *acts semisimply or nilpotently. Nor are the semisimple or nilpotent parts of the images in the image of the representation.*

Exercise 17 *If* $\mathfrak{g}$ *is nilpotent, its Killing form is identically 0.*

Exercise 18 *The radical of a Lie algebra need not equal the radical of its Killing form.*

Exercise 19 *Compute the Killing form of* $\mathfrak{sl}(2, \mathbb{F})$.