Workshop on Group Theory: Classification of Reductive Algebraic Groups
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Introduction to Lie Algebras

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Lecture 3
Inner Automorphisms, Solvable and Nilpotent Lie Algebras

Definition 1 The *center* of $\mathfrak{g}$ is defined by

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,Y] = 0 \ \forall Y \in \mathfrak{g}\} = \ker(\mathrm{ad}).$$

$Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. If $Z(\mathfrak{g}) = \mathfrak{g}$, then $\mathfrak{g}$ is abelian. If $Z(\mathfrak{g}) = \{0\}$, then ad is one-one and so $\mathfrak{g}$ is isomorphic to the linear Lie algebra $\mathrm{ad}(\mathfrak{g})$.

Definition 2 A Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals except itself and 0, and $[\mathfrak{g},\mathfrak{g}] \neq 0$. (The last requirement exactly excludes the one-dimensional Lie algebra!)

For example, $\mathfrak{sl}(2,\mathbb{F})$ is simple if $\mathrm{char}(\mathbb{F}) \neq 2$.

A simple Lie algebra must have zero center and hence the adjoint representation makes it isomorphic to a linear Lie algebra.

Definition 3 An *automorphism* of $\mathfrak{g}$ is an isomorphism with itself. The collection of all automorphisms of $\mathfrak{g}$ is denoted $\mathrm{Aut}(\mathfrak{g})$.

Example 4 Let $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$. Let $A \in L(V)$ be invertible. Then the map $X \mapsto AXA^{-1}$ is an automorphism of $\mathfrak{g}$. □

Until further notice, assume $\mathrm{char}(\mathbb{F}) = 0$**.**

Suppose $X \in L(V)$ is nilpotent: $X^M = 0$. Then we can define its exponential:

$$\exp(X) = \sum_{n=0}^{N-1} \frac{1}{n!} X^n.$$

Note that $\exp(X) \in L(V)$.

Exercise 5 *Suppose $X, Y \in L(V)$ are nilpotent and commute. Then*

$$\exp(X+Y) = \exp(X)\exp(Y).$$

In particular, $\exp(X)$ *has inverse* $\exp(-X)$.

Suppose $X \in \mathfrak{gl}(V)$ is nilpotent: $X^M = 0$. Let $l(X)$ denote left multiplication by X and $r(X)$ denote right multiplication by X. Then $l(X)$ and $r(X)$ are commuting nilpotent maps. In fact $l(X)^M = r(X)^M = 0$. Also, $\mathrm{ad}(X) = l(X) - r(X)$. Therefore

$$\mathrm{ad}(X)^{2M} = (l(X) - r(X))^{2M} = \sum_{n=0}^{2M} {}^{2M}C_n \, l(X)^n r(X)^{2M-n} = 0.$$

Thus $\mathrm{ad}(X)$ is nilpotent. So we can define $\exp(\mathrm{ad}(X)) : \mathfrak{g} \to \mathfrak{g}$ and it is a linear isomorphism. In fact,

$$\begin{aligned} \exp(\mathrm{ad}(X))Y &= \exp(l(X) - r(X))Y = \exp(l(X))\exp(-r(X))Y \\ &= \exp(l(X))Y\exp(-X) = \exp(X)Y\exp(-X) \end{aligned}$$

It is easy to see from this that $\exp(\mathrm{ad}(X)) \in \mathrm{Aut}(\mathfrak{gl}(V))$.

More generally, suppose D is a nilpotent derivation of $\mathfrak{g}$. Then $\exp(D)$ is defined and is a linear isomorphism of $\mathfrak{g}$. If $D^M = 0$, we calculate:

$$\begin{aligned} [\exp(D)X, \exp(D)Y] &= [\sum_{k=0}^{M} \frac{D^k}{k!}X, \sum_{l=0}^{M} \frac{D^l}{l!}Y] \\ &= \sum_{k=0}^{M}\sum_{l=0}^{M} [\frac{D^k}{k!}X, \frac{D^l}{l!}Y] \\ &= \sum_{n=0}^{2M}\sum_{i=0}^{n} [\frac{D^i}{i!}X, \frac{D^{n-i}}{(n-i)!}Y] \\ &= \sum_{n=0}^{2M} \frac{1}{n!} \sum_{i=0}^{n} {}^{n}C_i [D^i X, D^{n-i} Y] \\ &= \sum_{n=0}^{2M} \frac{1}{n!} D^n [X, Y] \qquad \text{(Leibniz Rule)} \\ &= \exp(D)[X, Y] \end{aligned}$$

Thus, $\exp(D) \in \mathrm{Aut}(\mathfrak{g})$. In particular, if $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)$ is nilpotent, then $\exp(\mathrm{ad}(X)) \in \mathrm{Aut}(\mathfrak{g})$.

Definition 6 $\mathrm{Int}(\mathfrak{g})$ denotes the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by automorphisms of the form $\exp(\mathrm{ad}(X))$, where $X \in \mathfrak{g}$ and $\mathrm{ad}(X)$ is nilpotent. Members of $\mathrm{Int}(\mathfrak{g})$ are called *inner* automorphisms.

Exercise 7 $\mathrm{Int}(\mathfrak{g})$ *is a normal subgroup of* $\mathrm{Aut}(\mathfrak{g})$*: If* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *and* $X \in \mathfrak{g}$ *then* $\varphi \exp(\mathrm{ad}(X))\varphi^{-1} = \exp(\mathrm{ad}(\varphi X))$

We now remove the assumption that $\mathrm{char}(\mathbb{F}) = 0$.

So far, we have developed the basic theory of Lie algebras without any restriction on their dimension (though our examples have been finite dimensional). From here on, we shall assume the Lie algebras to be finite dimensional (though certain infinite dimensional ones will temporarily appear later in this Workshop). And if $\mathfrak{g} \subset \mathfrak{gl}(V)$ is linear, we assume V is finite dimensional.

We shall now start exploring the structure of a Lie algebra via its ideals. On the one extreme, we have simple Lie algebras such as $\mathfrak{sl}(2, \mathbb{F})$ which have no non-trivial ideals. On the other, are the abelian ones in which every subspace is an ideal. In between are the algebras in the following example.

Example 8 Consider $\mathfrak{t} = \mathfrak{t}(n, \mathbb{F})$. Its commutator ideal is $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{n}(n, \mathbb{F})$, which is non-trivial. Every superspace of $\mathfrak{n}(n, \mathbb{F})$ is clearly an ideal in $\mathfrak{t}$. However, $\mathfrak{d}(n, \mathbb{F})$ is not an ideal.

Now let us consider $\mathfrak{n} = \mathfrak{n}(n, \mathbb{F})$. Its commutator ideal is

$$\mathfrak{n}^1 := [\mathfrak{n}, \mathfrak{n}] = \left\{ \begin{pmatrix} 0 & 0 & * & * \\ & \ddots & \ddots & * \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix} \right\}.$$

Moreover, $\mathfrak{n}^1$ is also an ideal of $\mathfrak{t}$: $[\mathfrak{n}^1, \mathfrak{t}] = \mathfrak{n}^1$. We can repeat these calculations using matrices where the non-zero entries keep shifting more and more towards the top-right corner. □

To bring order to these observations, we set up two series of nested ideals:

1. **Derived Series:** Define $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$, ..., and in general $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

2. **Lower (or Descending) Central Series:** Define $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1]$, ..., and in general $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$.

Exercise 9 *Prove that each* $\mathfrak{g}^{(i)}, \mathfrak{g}^i$ *is an ideal in* $\mathfrak{g}$*. (Hence each series is descending.)*

Exercise 10 *Let* $\mathfrak{h}$ *be an ideal in* $\mathfrak{g}$*. Show that each* $\mathfrak{h}^{(i)}, \mathfrak{h}^i$ *is an ideal in* $\mathfrak{g}$*.*

Definition 11 A Lie algebra $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(i)} = 0$ for some i. It is *nilpotent* if $\mathfrak{g}^i = 0$ for some i.

Clearly $\mathfrak{g}^i \supset \mathfrak{g}^{(i)}$ and so nilpotent Lie algebras are also solvable.

Exercise 12 *Show* $\mathfrak{t}(n, \mathbb{F})$ *is solvable but not nilpotent. On the other hand,* $\mathfrak{n}(n, \mathbb{F})$ *is nilpotent.*

Exercise 13 *If* $\mathfrak{g}$ *is solvable or nilpotent, then so is every subalgebra or homomorphic image of* $\mathfrak{g}$*.*

Exercise 14 *Let* $\text{char}(\mathbb{F}) = 0$*. Consider* $\mathfrak{sl}(2, \mathbb{F})$ *with the standard basis* (X, Y, H)*. Consider the inner automorphism defined by*

$$\sigma = \exp(\text{ad}X) \exp(-\text{ad}Y) \exp(\text{ad}X).$$

Show that σ *has the following action:*

$$H \mapsto -H, \quad X \mapsto -Y, \quad Y \mapsto -X.$$

Further, σ *is the same as conjugating by*

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 15 *If* $\dim(\mathfrak{g}) = 3$*,* $\mathfrak{g}$ *is either simple or solvable.*

Exercise 16 *If* $\mathfrak{g}$ *is nilpotent and non-zero then* $Z(\mathfrak{g}) \neq 0$*.*

Exercise 17 *The Lie algebra* $\mathfrak{g}$ *is semisimple iff it has no non-zero abelian ideals.*

Exercise 18 *If* $\text{char}(\mathbb{F}) = 2$ *then* $\mathfrak{sl}(2, \mathbb{F})$ *is nilpotent.*