Workshop on Group Theory: Classification of Reductive Algebraic Groups
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Introduction to Lie Algebras

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Lecture 2 - Ideals, Quotients & Homomorphisms

Let $\mathfrak{g}$ be a Lie algebra. Given $A, B \subset \mathfrak{g}$ we define their bracket by

$$[A, B] = \operatorname{span}\{[X, Y] : X \in A,\ y \in B\}.$$

Thus, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. It is an ideal if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Exercise 1 *Let $\mathfrak{h}_1, \mathfrak{h}_2$ be ideals of $\mathfrak{g}$. Then so are $\mathfrak{h}_1 \cap \mathfrak{h}_2$, $\mathfrak{h}_1 + \mathfrak{h}_2$ and $[\mathfrak{h}_1, \mathfrak{h}_2]$.*

Definition 2 The *derived algebra* of $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$. It is also called the *commutator ideal.*

Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$. Then the quotient vector space $\mathfrak{g}/\mathfrak{h}$ becomes a Lie algebra under the bracket

$$[X + \mathfrak{h}, Y + \mathfrak{h}] := [X, Y] + \mathfrak{h}.$$

Exercise 3 *Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$. The quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, $X \mapsto X + \mathfrak{h}$ is a Lie algebra homomorphism.*

One has the usual results about ideals and quotients:

Exercise 4 *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then $\mathfrak{g}/\ker\varphi$ is isomorphic to $\operatorname{im}\varphi$.*

Exercise 5 *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism, and $\mathfrak{j}$ an ideal of $\mathfrak{g}$ contained in $\ker\varphi$. Then there is a unique homomorphism $\psi : \mathfrak{g}/\mathfrak{j} \to \mathfrak{h}$*

such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[d, "\pi"] \arrow[dr, "\wp"] & \\
\mathfrak{g}/\mathfrak{j} \arrow[r, "\psi"] & \mathfrak{h}
\end{tikzcd}$$

Exercise 6 *If* $\mathfrak{j}, \mathfrak{k}$ *are ideals of* $\mathfrak{g}$ *such that* $\mathfrak{j} \subset \mathfrak{k}$*, then* $\mathfrak{k}/\mathfrak{j}$ *is an ideal in* $\mathfrak{g}/fj$ *and*

$$\frac{\mathfrak{g}/\mathfrak{j}}{\mathfrak{k}/\mathfrak{j}} \textit{ is naturally isomorphic to } \mathfrak{g}/\mathfrak{k}.$$

Exercise 7 *If* $\mathfrak{j}, \mathfrak{k}$ *are ideals of* $\mathfrak{g}$*, then*

$$\frac{\mathfrak{j}+\mathfrak{k}}{\mathfrak{j}} \textit{ is naturally isomorphic to } \frac{\mathfrak{k}}{\mathfrak{j}\cap\mathfrak{k}}.$$

Example 8 As the Workshop proceeds, we shall see that the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ has a special role to play. Let us look at its structure in detail. A natural choice of basis for $\mathfrak{g}$ is:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the bracket is bilinear, we only have to understand the bracket relations between these basis elements. They turn out to have a simple form:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

In fact, consider the linear $\mathfrak{g} \to \mathfrak{g}$ map defined by $Z \mapsto [H, Z]$. The map is diagonalizable and the basis elements are its eigenvectors! This suggests that it would be useful to study the bracket via the linear maps it induces.

An implication of these calculations is that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ if $\mathrm{char}(\mathbb{F}) \neq 2$. $\square$

Definition 9 Let $\mathfrak{g}$ be a Lie algebra. For any $X \in \mathfrak{g}$, define a linear map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}(X)Y = [X, Y].$$

This map is called the *adjoint of* X.

Exercise 10 *Show that* $\mathrm{ad}(X) = \mathrm{ad}(Y)$ *implies* $[X, Y] = 0$*. Is the converse true?*

Exercise 11 *Show that* $\mathrm{ad}(X)[Y,Z] = [\mathrm{ad}(X)Y, Z] + [Y, \mathrm{ad}(X)Z]$.

Definition 12 A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is a *derivation* if

$$D[X,Y] = [DX,Y] + [X,DY], \qquad \forall X, Y \in \mathfrak{g}.$$

The collection of all derivations of $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$.

Exercise 13 *Show that any derivation* D *of* $\mathfrak{g}$ *satisfies the* Leibniz rule*:*

$$D^n[X,Y] = \sum_{i=0}^{n} {}^nC_i \, [D^i X, D^{n-i} Y].$$

Remark The binomial coefficients are defined for arbitrary $\mathbb{F}$ recursively by

$${}^nC_0 = {}^nC_n = 1 \quad \text{and} \quad {}^nC_i = {}^{n-1}C_{i-1} + {}^{n-1}C_i \text{ for } 0 < i < n.$$

Exercise 14 *Show that* $\mathrm{Der}(\mathfrak{g})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{g})$.

Exercise 15 *Show that the* adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a representation of* $\mathfrak{g}$ *in* $\mathfrak{g}$.

Note that the image of the adjoint representation lies in $\mathrm{Der}(\mathfrak{g})$. Members of this image are called *inner derivations*. Derivations which are not inner are called *outer*.

Exercise 16 $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ *is an abelian Lie algebra.*

Exercise 17 *Show that* $\mathfrak{t}(n,\mathbb{F}) = \mathfrak{d}(n,\mathbb{F}) + \mathfrak{n}(n,\mathbb{F})$ *(vector space direct sum). Also:*

$$\begin{aligned} [\mathfrak{d}(n,\mathbb{F}), \mathfrak{n}(n,\mathbb{F})] &= \mathfrak{n}(n,\mathbb{F}) \\ [\mathfrak{t}(n,\mathbb{F}), \mathfrak{t}(n,\mathbb{F})] &= \mathfrak{n}(n,\mathbb{F}) \end{aligned}$$

Exercise 18 *Let* $X \in \mathfrak{gl}(n,\mathbb{F})$ *be diagonalizable with eigenvalues* $a_1, \ldots, a_n$. *Then* $\mathrm{ad}(X)$ *is diagonalizable with eigenvalues* $a_i - a_j$ $(1 \le i, j \le n)$.

Exercise 19 *Show that the center of* $\mathfrak{gl}(n,\mathbb{F})$ *equals* $\mathfrak{s}(n,\mathbb{F})$, *which consists of the scalar matrices. In addition,*

$$\begin{aligned} \mathrm{char}(\mathbb{F}) \nmid n \quad &\Longrightarrow \quad \mathfrak{s}(n,\mathbb{F}) + \mathfrak{sl}(n,\mathbb{F}) = \mathfrak{gl}(n,\mathbb{F}) \\ &\qquad\quad \textit{(vector space direct sum)}, \\ \mathrm{char}(\mathbb{F}) \neq 2 \quad &\Longrightarrow \quad [\mathfrak{sl}(n,\mathbb{F}), \mathfrak{sl}(n,\mathbb{F})] = \mathfrak{sl}(n,\mathbb{F}), \\ 2 \neq \mathrm{char}(\mathbb{F}) \nmid n \quad &\Longrightarrow \quad [\mathfrak{gl}(n,\mathbb{F}), \mathfrak{gl}(n,\mathbb{F})] = \mathfrak{sl}(n,\mathbb{F}). \end{aligned}$$

Exercise 20 *If* $\operatorname{char}(\mathbb{F}) \nmid n$ *then the center of* $\mathfrak{sl}(n, \mathbb{F})$ *is 0. Else, it is* $\mathfrak{s}(n, \mathbb{F})$.

Exercise 21 *If* $\operatorname{char}(\mathbb{F}) \neq 2$ *then the only non-trivial ideals in* $\mathfrak{gl}(2, \mathbb{F})$ *are* $\mathfrak{sl}(2, \mathbb{F})$ *and* $\mathfrak{s}(2, \mathbb{F})$.

Exercise 22 *Let* $X \in \mathfrak{g}$ *and* $D \in \operatorname{Der}(\mathfrak{g})$. *Then* $[D, \operatorname{ad}(X)] = \operatorname{ad}(DX)$. *(Hence the inner derivations form an ideal in* $\operatorname{Der}(\mathfrak{g})$.*)*

Exercise 23 *Show that* $\mathfrak{sl}(2, \mathbb{F})$ *has no non-trivial ideals if* $\operatorname{char}(\mathbb{F}) \neq 2$.