# Mathematics & Finance

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**Abstract**

Finance is the study of how to invest money. The basic premise is that to make a profit, one must undertake risk. The greater the desired profit, the more the associated risk.

In the last 30 or so years, there has been tremendous growth in the use of mathematics to quantify this relationship of profit with risk. The mathematics involved ranges from elementary to advanced, and the needs of Finance have also led to advances in mathematical techniques.

We will look at two applications of Mathematics to Finance, including work that has received the Nobel Prize.

## 1 Introduction

Mathematics has always existed in a close relationship with other areas of human endeavour such as science, technology, and religion. Occasionally it has also interacted with art, poetry, and literature. One relationship which has been persistently strong is the one with financial matters. Early developments in arithmetic owed much to the needs of accounting, and even geometry was influenced by the need of the state to measure area to fix taxes.

While Economics deals with the general issues regarding money and its place in society, Finance has a more narrow aim: How should we invest our money to make it grow faster? This narrower focus simplifies the situation and makes it more tractable to mathematical treatment.

Naturally, we would like to invest in assets whose value seems likely to increase at a faster rate. It is almost a law of Nature, however, that bigger promises are also less reliable. In fact, less reliable promises *must* be bigger, if they are to have any takers. Thus there is a trade-off between expected profit and risk: to aim for higher profit, the investor must undertake greater risk.

The word risk is used in Finance in a special way. It refers to uncertainty, and does not necessarily have a negative connotation. Thus, consider the choice between putting money in a bank account or using it to buy shares in a company. The second investment is riskier because it has more uncertainty, but it is not obvious how its worth compares to that of the the first one.

This example leads us to the role of Probability in Finance. The relative worth of the second choice depends on the probabilities of the possible payoffs. If higher payoffs are perceived as more likely, its value will increase. If we can model the fluctuations in stock prices, we can assign probabilities to the possible payoffs from this investment, and thus estimate its value to the investor.

In the main body of this essay, we will first review the standard model for fluctuations of stock prices. Then we will see how it can be applied to pricing certain contracts about future trades.

# 2 Modeling Stock Prices

The first mathematical model for fluctuating prices was created by Louis Bachelier in 1900 who showed, for example, that it could explain the price fluctuations of French government bonds. Bachelier was a student of Henri Poincaré who, along with David Hilbert, was the pre-eminent mathematician of his time. Unfortunately, Bachelier's thesis *Théorie de la Spéculation* did not attract much attention.

The task was next taken up by Paul Samuelson in 1965. His model of **Geometric Brownian Motion** or **GBM** is still the standard one. Mathematically, GBM makes a small adjustment to Bachelier's model which significantly improves its properties.

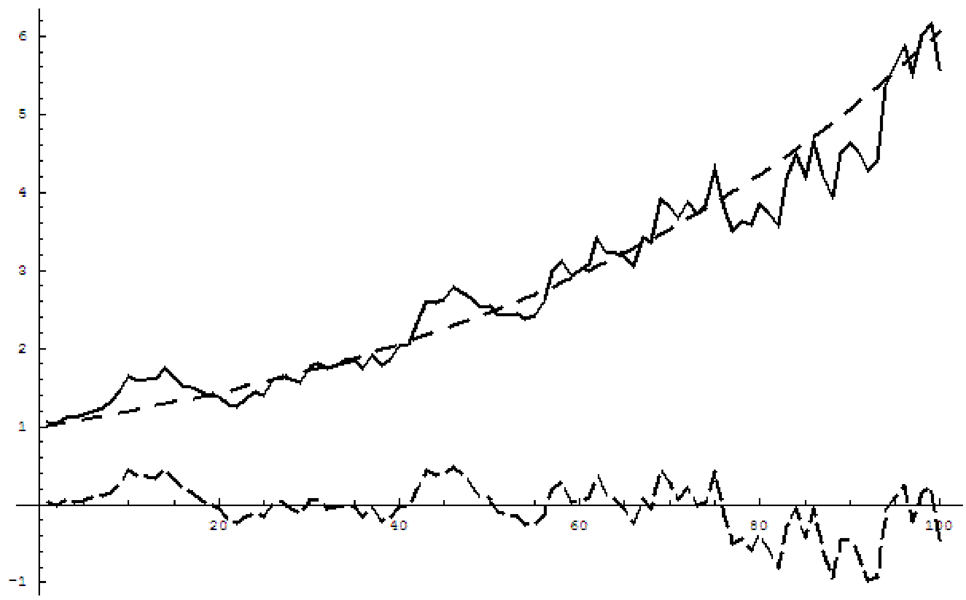


Figure 1: Decomposition of a typical stock price chart (solid) into drift and volatility (dashed).

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GBM treats stock prices as made of two parts: a steady growth called **drift** combined with random fluctuations called **volatility** (See Figure 1). The volatility is centered around zero and is modelled by the normal distribution.

Let us list the main features of GBM:

- Assumes amount of drift and volatility is constant over time.
- Beyond this, future behaviour is independent of past behaviour. Past trends are not relevant to predicting the future!
- The model is based on the normal distribution, hence up and down moves are likely to be of the same size, and large changes are unlikely.

The model works well for most stocks most of the time. Figure 2 illustrates this by comparing a computer simulation of GBM with the actual behaviour of the BSE Sensex index. Since the model has an element of randomness, it does not attempt to recreate the exact path of the actual prices. However, it shows what kind of behaviour can be expected. Notice that the amount

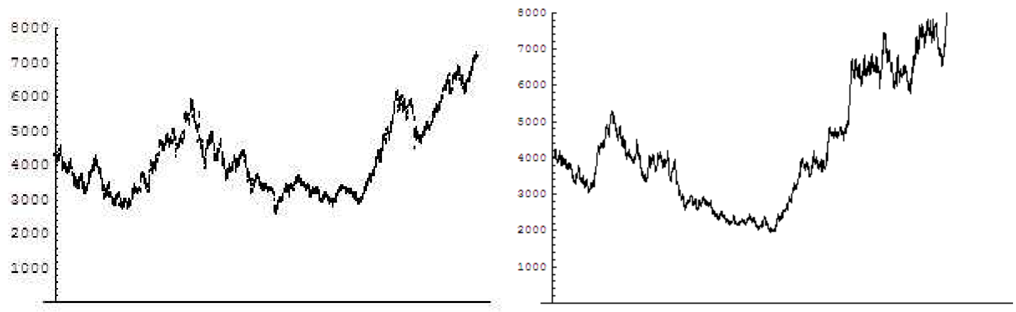


Figure 2: The first chart presents the daily closing prices of the BSE Sensex index over 8 years from July 1997 to July 2005. The second chart is a simulation of GBM with the same drift and volatility.

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of daily fluctuation is just about the same in the model and in reality. In the longer run too, the way up and down trends come into being is similar. Thus, it is reasonable to make probabilistic predictions on the basis of GBM.

Nevertheless, the real world differs from that of GBM in important ways (See Figure 3):

- Large down moves are more likely than large up moves.
- Large moves are more common than the model predicts.

To make GBM a better fit to the real world, we have to replace the normal distribution by distributions which are asymmetric and heavy-tailed (i.e., large values are reasonably likely). These are mathematically difficult to handle (for instance, their variance is undefined and so the standard techniques of Statistics cannot be used), and at present are best dealt with using computer simulations. They are needed, however, in a wide variety of other applications such as:

- The determining of insurance premiums.
- The prediction of natural disasters.

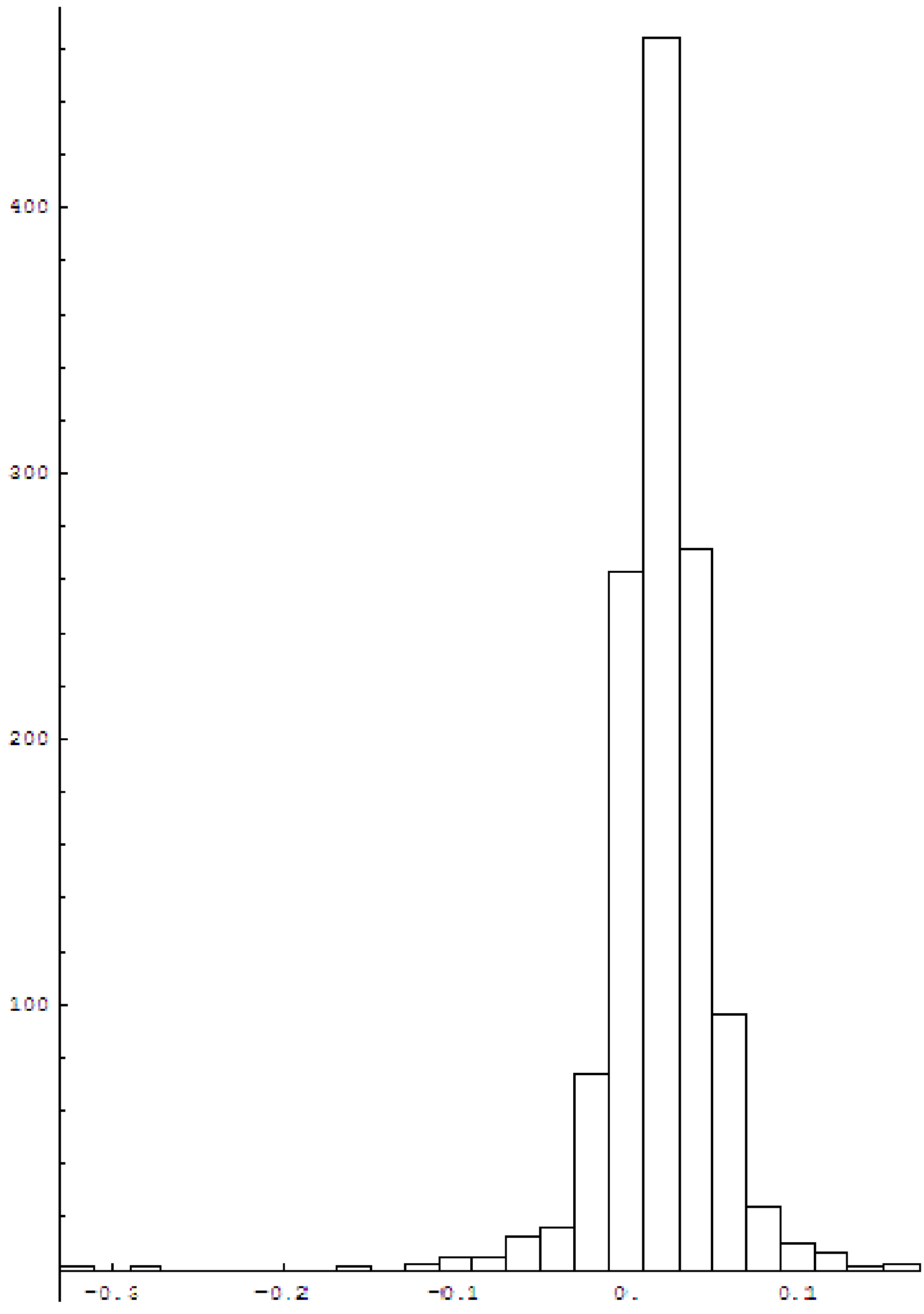


Figure 3: Daily changes in Infosys stock prices during 2000-5. While the shape is roughly symmetric, there are some large falls corresponding to crashes in July 2004 and April 2003 but no corresponding jumps on the other side.

# 3 Financial Derivatives

A **derivative** is a contract about a **future** trade. Consider this example:

A breakfast cereal company and a farmer will trade in wheat 3 months from "now", after the harvest. There are two possibilities that could worry the parties:

- Poor crop $\implies$ prices rise $\implies$ loss for the company.
- Bumper crop $\implies$ prices fall $\implies$ loss for the farmer.

Both parties can eliminate their risk by agreeing *now* on what price they will trade in 3 months time. Such an agreement, called a **futures contract**, is the most basic kind of derivative.

In eliminating risk, a futures contract also eliminates the possibility of benefiting from favourable circumstances. Thus, the farmer would prefer a contract where:

- The company promises to buy a certain amount at a certain price from her in 3 months.
- She has the option of cancelling the contract if the market price after 3 months is higher than the contract price.

Such contracts exist and are called **call options**. Now, a basic feature of Finance is that **everything has a price**. In particular, in a call option, the right to drop out of the contract has to be bought – our farmer must make an initial small payment to the cereal company.

The two parties in a call option are called the holder and the writer. The structure of the contract is as follows:

1. The holder pays the writer a fee (**call premium**) to buy the contract.
2. On the **expiration date** $T$, the holder *may* pay the writer an amount $X$ called the **exercise price**.

3. *If* the holder pays up, the writer *must* deliver the underlying asset.

Futures have existed in the U.S. since 1848, but until about 30 years ago there was **no** theoretical understanding of the “fair” value of the call premium and hence no large-scale use of call options.

The breakthrough came in 1973 with the publication of the joint work of Fischer Black and Myron Scholes as well as the work of Robert Merton. They made the following simplifying assumptions:

- The asset price follows GBM.
- Trading takes place continuously.
- There is no income (such as dividends) from the asset.
- Interest rates are constant over the life of the contract.
- Trading in th eunderlying asset involves no transaction costs.
- **No-Arbitrage Principle:** Every profit (greater than current interest rates) is risky.

We should mention here that the No-Arbitrage Principle is a surprisingly powerful tool for finding the fair price of an asset. Indeed, it underlies almost every application of mathematics to Finance.

Based on these assumptions, Black-Scholes-Merton derived the following formula for the call premium $C$:

$$\boxed{C = S\Phi(w) - Xe^{-rT}\Phi(w - \sigma\sqrt{T})}$$

where

$$\boxed{w = \frac{(r + \sigma^2/2)T - \ln(X/S)}{\sigma\sqrt{T}}}$$

and

$\Phi$ = CDF of the standard normal distribution, $r$ = current interest rate, $S$ = current price of the asset, and $\sigma$ measures the volatility of the asset.

The Black-Scholes-Merton formula (in the literature it is usually just called the Black-Scholes formula) has various applications:

- Pricing options.
- Analysis of the dependence of call premium on factors such as interest rates and asset price, and using this to create portfolios which are less vulnerable to fluctuations in these factors.
- Probabilistic estimates of possible gain or loss from investing in options.
- Calculating implied volatility (see below).

The volatility of an asset price can be estimated from the past fluctuations in it, but there are two difficulties:

- The value obtained depends on the time window used.
- Why should future volatility be given by past volatility?

An alternative is to put the observed call premium into the Black-Scholes-Merton formula, treat volatility ($\sigma$) as the only unknown and solve for it. This is called its **implied volatility**, and is now considered to be a more reliable indicator of future behaviour. The usefulness of implied volatility gives indirect support to the correctness of the Black-Scholes-Merton model.

# 4 The Maths Involved

The topics we have considered can be explored using mathematics at the level of high school or elementary undergraduate studies. All that is needed is:

- Calculus (partial derivatives, integration)
- Probability (normal distribution, independent random variables, expectation and variance)

Combined with basic principles of Finance such as the No-Arbitrage Principle, these are enough to motivate GBM and the Black-Scholes Formula.

However, modern Finance goes far beyond these topics and deals with considerably more complicated situations and questions. Current research into Finance involves mathematical inputs from a diverse range of areas:

- Measure Theory
- Stochastic Differential Equations
- Monte Carlo Techniques
- Game Theory

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