A Characteristic-Based Approximate Riemann Solver

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Abstract

An efficient non-iterative approximate Riemann solver is proposed. It is based on linearized differential relations along characteristic curves of the one-dimensional hyperbolic conservation laws. The high performance of this solution procedure is demonstrated on 1D shock tube test cases.

Introduction

For the class of finite-volume difference schemes for systems of hyperbolic conservation laws a special solution procedure is used to compute flux vector at the cell interface in accordance with initial discontinuous data and the hyperbolic nature of the governing equations. This procedure, often referred to as Riemann solver, is usually interpreted as a technique for obtaining solution to the certain initial value problem (IVP) with discontinuous, in general, initial conditions. This solution (one- or multidimensional, exact or approximate) is often intrpreted as an evolution in time of the initial distribution. Another possible interpretation of the IVP is that the computed parameters at interface must only be consistent with the system of characteristic relations, which for 1D case take form of the relations along characteristic curves. The linearized characteristic relations can be used since it is known that solvers based on linearizations (e. g. [2]) don't affect notably the numerical results produced by the conservative numerical schemes.

The construction of the flux evaluation procedure from the set of scalar characteristic fields may produce ultimate economical Riemann solver. Such a solver is described below.

Description of the Riemann solver

Let us step-by-step outline this solution procedure for compressible fluid dynamics.

1. Consider two states: $\overline{U_R}$ and $\overline{U_L}$.

2. Reduce the problem to 1D considering only normal components of velocity (needed for 2D and 3D).

3. For the 1D conservation laws for compressible fluid dynamics

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial \overline{F}}{\partial x} = 0 \tag{1}$$

where $\overline{U} = [\rho, \rho u, \rho E]^T, \overline{E} = [\rho u, \rho u^2 + p, \rho u E + pu]^T, E = e(\rho, p) + 0.5u^2$, exact characteristic relations (differences along characteristic directions), for almost arbitrary equation of state, have form

$$\left\{ \begin{array}{l} d^- u - \frac{1}{\rho c} d^- p = 0 \\ d^o p - c^2 d^o p = 0 \\ d^+ u + \frac{1}{\rho c} d^+ p = 0 \end{array} \right. \tag{2}$$

where corresponding characteristic directions in (x,t)-plane are defined by:

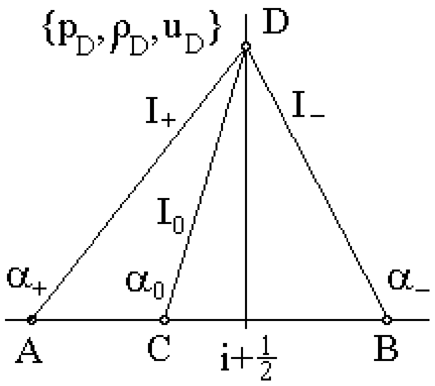

Figure 1: The characteristic stencil for $(u-c)<0$, $u>0$, $(u+c)>0$.

$$\begin{cases} d^- x = (u-c)d^- t \\ d^o x = u d^o t \\ d^+ x = (u+c)d^+ t \end{cases}$$

Linearization of (2) gives the following partial differential equations for "invariants":

$$\begin{cases} d^- I_- = 0 \\ d^o I_o = 0 \\ d^+ I_+ = 0 \end{cases} \tag{3}$$

where "invariants" are $I_- = u + \alpha_- p$, $I_o = p + \alpha_o \rho$, $I_+ = u + \alpha_+ p$, and $\alpha_- = -1/\rho c$, $\alpha_o = -c^2$, $\alpha_+ = 1/\rho c$ are constant coefficients. Though relations (3) are valid only for smooth solutions and for infinitely small disturbances soince α_-, α_o, α_+ are constants, one still may try to use (3) to obtain approximate solution to "finitely disturbed" IVP.

4. For clarity let us consider the case $(u-c)<0$, $u>0$, $(u+c)>0$ (shown on Fig. 1) and determine "invariants" arriving at point D using (3) and some parameters from points A, B, C:

$$I_{-D} = u_B + \alpha_- p_B,$$
$$I_{oD} = p_C + \alpha_o \rho_C,$$
$$I_{+D} = u_A + \alpha_+ p_A.$$

Hereby a linearization of another kind is introduced: these "constants" are taken from different points around the initial discontinuity, e.g. in a simplest way: $\alpha_- = \alpha_B$, $\alpha_o = \alpha_C$, $\alpha_+ = \alpha_A$. Though points A, B, C and D are infinitely close to each other in space and in time, the corresponding flow parameters may be different, so additional linearization obviously takes place.

This [simplest] method of evaluation of constants α_-, α_o, α_+ should be preferred for subsonic flows, but also it is possible to evaluate these constants e. g. in such way: $\alpha_+ = (\alpha_{+A} + \alpha_{+D})/2$, etc. Solver of the proposed type that uses such "weaker" linearization will produce flux vector rather close to that for exact solution.

The concern, however, is that both the invariants and the constants must be evaluated using values from R or L states according to signs of characteristic speeds $(u-c)$, u and $(u+c)$.

5. Calculate parameters at the cell interface (point D, see Fig. 1) by "decoding" the "invariants". With indices D dropped, we have:

$$p = \frac{I_+ - I_-}{\alpha_+ - \alpha_-}, \quad u = I_+ - \alpha_+ p, \quad \rho = \frac{I_o - p}{\alpha_o}, \quad c = c(p, \rho).$$

6. (Optionally) check signs of $(u-c)$, u and $(u+c)$. If they are the same as those initially taken, the procedure is over. Otherwise it may be reasonable to return to step 4. In this case it is better to evaluate constants in a "weakly linearized" fashion: $\alpha_+ = (\alpha_{+A} + \alpha_{+D})/2$, etc.

7. Use tangential velocity due to flow direction (+/-) normal to the interface to evaluate tangential momentum and kinetic energy fluxes (this is needed for 2D or 3D).

The idea of this approximate solution procedure was inspired by the article [1], which proposed to blend these relations for "invariants" and the exact Riemann solver of Godunov. Hopefully the proposed solver is the fastest possible solver based on the hyperbolic properties of the governing equations (1), (except for solvers derived from pure acoustics)!

Exact relations along characteristics may also be used in flux evaluation procedures, but for discontinuous IVPs a certain linearization is still needed: some coefficients still have to be chosen from the pointwise values.

Test problems

Let us consider the simple shock-tube problem

$$\rho(x,0) = p(x,0) = \begin{cases} 2, x \leq 0 \\ 1, x > 0 \end{cases}, u(x,0) = 0, \tag{4}$$

governed by the 1D Euler equations (1) for the compressible fluid with the equation of state of the perfect gas with $\gamma = \frac{c_p}{c_v} = 1.4$. Exact solution to this problem contains subsonic self-similar flow with shock wave, contact discontinuity and rarefaction wave. Two related Godunov-type schemes that use approximate linearized Riemann solver described above were used to obtain numerical solution to the problem (4). The first scheme is a first-order Godunov scheme

$$\overline{U_i}^{n+1} = \overline{U_i}^n + \Delta t \cdot L(\overline{U_i}^n), \tag{5}$$

where

$$L(\overline{U_i}^n) = \frac{1}{\Delta x}(\overline{F}^n_{i-\frac{1}{2}} - \overline{F}^n_{i+\frac{1}{2}}).$$

The second one is the three-step Runge-Kutta scheme [3]:

$$\begin{aligned} \overline{U}_i^{(1)} &= \overline{U_i}^n + \Delta t \cdot L(\overline{U_i}^n), \\ \overline{U}_i^{(2)} &= \tfrac{3}{4}\overline{U_i}^n + \tfrac{1}{4}[\overline{U_i}^{(1)} + \Delta t \cdot L(\overline{U_i}^{(1)}), \\ \overline{U_i}^{n+1} &= \tfrac{1}{3}\overline{U_i}^n + \tfrac{2}{3}[\overline{U_i}^{(2)} + \Delta t \cdot L(\overline{U_i}^{(2)}), \end{aligned} \tag{6}$$

where parameters at cell interfaces are defined by the piecewise-parabolic spatial reconstruction of solution within each cell:

$$\begin{aligned} \overline{F}^n_{i+1/2} &= \overline{F}(\overline{U}^n_{i+1/2,L}, \overline{U}^n_{i+1/2,R}), \\ \overline{U}^n_{i+1/2,L} &= \overline{U}^n_i + [S^{-1}]^n_i \cdot \left\{ \tfrac{1+\varphi}{4}\widetilde{\Delta}\overline{W}^n_i + \tfrac{1-\varphi}{4}\widetilde{\nabla}\overline{W}^n_i \right\}, \\ \overline{U}^n_{i+1/2,R} &= \overline{U}^n_{i+1} + [S^{-1}]^n_{i+1} \cdot \left\{ \tfrac{1+\varphi}{4}\widetilde{\nabla}\overline{W}^n_{i+1} + \tfrac{1-\varphi}{4}\widetilde{\Delta}\overline{W}^n_{i+1} \right\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \widetilde{\Delta}\overline{W}^n_i &= minmod(\Delta\overline{W}^n_i, b \cdot \nabla\overline{W}^n_i), \quad \Delta\overline{W}^n_i = [S]^n_i \cdot \Delta\overline{U}^n_i, \\ \widetilde{\nabla}\overline{W}^n_i &= minmod(\nabla\overline{W}^n_i, b \cdot \Delta\overline{W}^n_i), \quad \nabla\overline{W}^n_i = [S]^n_i \cdot \nabla\overline{U}^n_i, \end{aligned}$$

with matrices taken from the following transform:

$$[A] = [S^{-1}][L][S], \quad [L] = diag(\lambda_1, \lambda_2, \lambda_3) = diag(u, u+c, u-c),$$

for the non-conservative form of (1):

$$\frac{\partial \overline{U}}{\partial t} + [A]\frac{\partial \overline{U}}{\partial x} = 0.$$

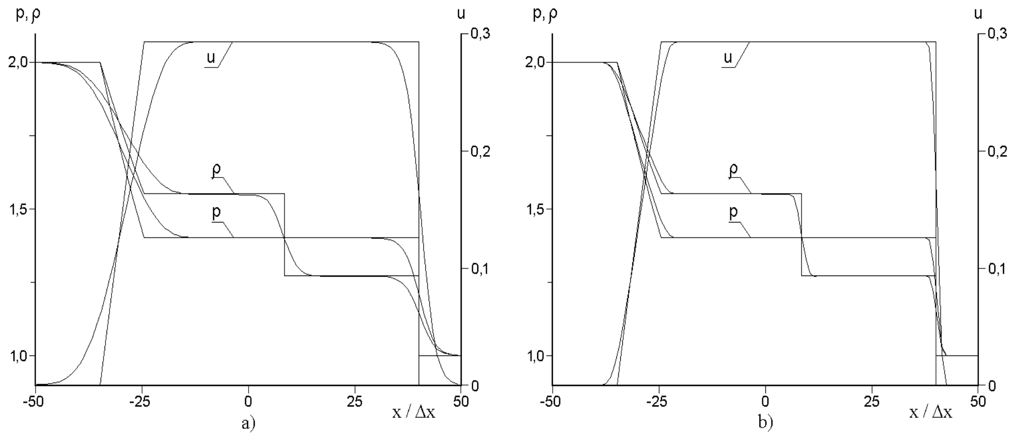

Figure 2: Numerical solutions to the "subsonic" shock-tube problem (4) given by two Godunov-type schemes: (5) and (6), compared with the exact solution.

and with the limiter function $minmod(\cdot,\cdot)$ taken from [4].

Simutions by these schemes were carried out using fixed Courant number $\Delta t/(\Delta x \cdot max(|u| + c)) = 0.5$ on the grid with 100 equally spaced cells for 100 steps in time. The numerical solutions to (4) are compared to the exact solution on Fig. 2 a) and b) for the first-order (5) and for the higher order (6) scheme respectively. As one can see, the use of proposed linearized procedure does not affect the quality of the numerical solution. For the first-order scheme with classical solver of Godunov the execution time was about 4.2 times greater than for the case of scheme with the proposed approximate solver.

Another test problem (with much stronger initial discontinuity of pressure and density)

$$\rho(x,0) = p(x,0) = \begin{cases} 20, x \leq 0 \\ 1, x > 0 \end{cases}, u(x,0) = 0, \tag{8}$$

was solved numerically using three-step Runge-Kutta scheme (6) with limited piecewise-parabolic reconstruction (7). Solution to this problem contains zone of supersonic flow (between the contact discontinuity and the rarefaction wave) with $M = 1.356$. Numerical simulation was done using iterative version of the proposed Riemann solver, that turned out to be necessary for proper handling supersonic flow. Shown on Fig. 3 is the numerical solution obtained on the grid with 100 cells after 100 steps in time with the Courant number 0.55 is compared with the exact solution.

Discussion

The proposed approximate linearized Riemann solver is essentially non-iterative and does not require vector/matrix multiplication. It is extremely fast and as it is usually the case, the linearized solver works well and the resulting scheme can produce quite satisfactory numerical solutions for the sub/supersonic problems governed by hyperbolic conservation laws of gas dynamics.

No difficulties encountered when this procedure was implemented in different 1-, 2-, or 3-dimensional Godunov-type schemes. For large-scale computations this solver can help to save considerable amounts of computer time. And it must be noted that the fluxes computed by this linearized procedure are rather close to those given by classical Roe's [2]. This fact leads to conclusion that both these approaches are intimately related.

Obviously in this solver the resulting flux vector $\overline{F}(\overline{U}_L, \overline{U}_R)$ does not depend continuously on $\overline{U}_L$and $\overline{U}_R$ because of switching between characteristic stencils, in transonic regions, in particular.

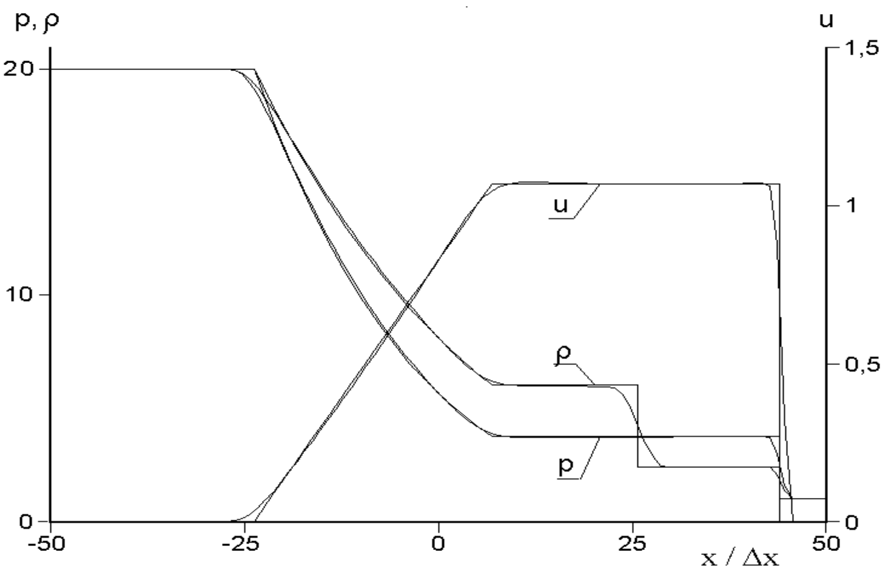

Figure 3: Solution to the "supersonic" Riemann problem (8) given by three-step scheme (6) compared to the exact solution.

References

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[4] Chakravarthy S.R, Osher S. A new class of high-accuracy TVD schemes for hyperbolic conservation laws. // AIAA Paper 85-0363. 1985 - 11 p.

The updated version of this dicument can be found at
http://www.geocities.com/andrei_chernousov/CODES/solver.ps.gz. See also: godunov.tar.gz, lcs.tar.gz, lcs3.tar.gz (in the same directory).