Theorems Involving Multidimensional Laplace Transforms

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Introduction

For any SISO analytic system, there exists a functional series representation of the form

(1)
$$
\begin{aligned}
y(t) &= h_o + \int_{-\infty}^{+\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \\
&+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \\
&\vdots \\
&+ \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n(\tau_1,\tau_2,\cdots,\tau_n)x(t-\tau_1)x(t-\tau_2)\cdots x(t-\tau_n)d\tau_1\cdots d\tau_n
\end{aligned}
$$

[1]

where y is the output, x is the input and $h_i(\tau_1,\tau_2,\cdots,\tau_i)$ are the kernels of the system. For linear systems, all kernels except $h_1(t)$ are zero. For linear time-varying systems, h_2 exists also.

We have the multidimensional Laplace transform pair,

(2) $$\mathrm{F}(\bar{s}) = L_n\{\mathrm{f}(\bar{t}); \bar{s}\} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \mathrm{f}(\bar{t})\mathrm{e}^{(-\bar{s}\bullet\bar{t})}\mathrm{P}_n(d\bar{t})$$

(3) $$\mathrm{f}(\bar{t}) = L_n^{-1}\{\mathrm{F}(\bar{s}); \bar{t}\} = \left(\frac{1}{2\pi j}\right)^n \int_{\sigma_n - j\infty}^{\sigma_n + j\infty}\int_{\sigma_{n-1} - j\infty}^{\sigma_{n-1} + j\infty}\cdots\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty}\mathrm{F}(\bar{s})\mathrm{e}^{(\bar{s}\bullet\bar{t})}\mathrm{P}_n(d\bar{s})$$

Where $\bar{t} = (t_1, t_2, \cdots, t_n)$, $\bar{s} = (s_1, s_2, \cdots, s_n)$, $\bar{s}\bullet\bar{t} = \sum_{i=1}^{n} s_i t_i$, $\mathrm{P_n}(d\bar{x}) = \prod_{i=1}^{n} dx_i$.

Part I - An “Association Reduction” Method [1]

Applying the multidimensional Laplace transform to equation (1), neglecting the trivial first term, yields

(4)
$$
\begin{aligned}
\mathrm{Y}(s_1) &= \mathrm{H}(s_1)\mathrm{X}(s_1) + \mathrm{R}_1\{\mathrm{H}_2(s_1,s_2)\mathrm{X}(s_1)\mathrm{X}(s_2)\} + \cdots \\
&+ \mathrm{R}_{n-1}\{\mathrm{H}_n(s_1,s_2,\cdots,s_n)\mathrm{X}(s_1)\mathrm{X}(s_2)\cdots\mathrm{X}(s_n)\}
\end{aligned}
$$

Where $\mathrm{R_i}$ is called an “i^{th} order association reduction”, defined to be

(5)
$$
\begin{aligned}
\mathrm{R_i}\{\mathrm{F}_n(\bar{s})\} &\stackrel{\Delta}{=} \left(\frac{1}{2\pi j}\right)^i \int_{\sigma_{n-1+1} - j\infty}^{\sigma_{n-i+1} + j\infty}\cdots\int_{\sigma_n - j\infty}^{\sigma_n + j\infty} dU_n dU_{n-1}\cdots dU_{n-i+1} \\
&\bullet \mathrm{F}_n(\bar{s})\Big|_{\substack{s_n = U_n \\ s_{n-1} = U_{n-1} - U_n \\ \vdots \\ s_{n-i+1} = U_{n-i+1} - U_{n-i+2} \\ s_{n-i} = U_{n-i} - U_{n-i+1}}} \\
&= \mathrm{F_n}(s_1, s_2, \cdots, s_{n-i}),
\end{aligned}
$$

where $1 \le i \le n-1$.

It has been shown [2] any kernel in a system composed of linear time-varying subsystems and time-invariant nonlinearities represented by a power series and combined by the operations multiplication, addition, cascade and feedback, can be written in the form

$$\text{(6)} \qquad \mathrm{H_n}\left(\bar{s}\right) = \sum_{i=1}^{n} \left[\frac{\mathrm{N_n}\left(\bar{s}\right)}{\prod_{k} \left(\sum_{i=1}^{n} s_i \right) + \delta_k} \right],$$

where the $\mathrm{N_n}\left(\bar{s}\right)$ are polynomials in n variables. Some of the n variables can be missing, but not so as to create overlapping pole terms. "Overlapping" means that two terms contain a common complex variable while each contains at least one complex variable not contained in the other. For example, the terms $(s_1 + s_3 + \alpha)$ and $(s_1 + s_2 + \gamma)$ are overlapping since each contains s_1 and only the second, s_2. Clearly $(s_1 + s_2 + s_3 + \alpha)$ and $(s_1 + s_2 + \gamma)$ are not overlapping.

A Heaviside-type expansion can be used to reduce and expand multidimensional Laplace transform kernels and thus eliminate the need to evaluate the integrals of the association reduction formula (5).
Applying this to two-dimensional kernels, they can be reduced to one s-variable and then solved by traditional techniques.

Reduction and expansion of two-dimensional kernels
The partial fractional expansion can be most easily introduced through its application to two-dimensional kernels.
It is desired to reduce the general two-dimensional kernel

$$\text{(7)} \qquad \mathrm{F_2}(s_1, s_2) = \frac{\mathrm{N_2}(s_1, s_2)}{\prod_{i=1}^{II} (s_1 + s_2 + \alpha_i) \prod_{j=1}^{JJ} (s_1 + \beta_j) \prod_{k=1}^{KK} (s_2 + \gamma_k)}$$

Applying the association reduction (5) to this kernel,

$$\mathrm{F_2}(s) = \mathrm{R_1}\{\mathrm{F_2}(s_1, s_2)\}$$

$$\text{(8)} \qquad = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \left\{ \frac{\mathrm{N_2}(s_1, s_2)}{\prod_{i=1}^{II} (s_1 + s_2 + \alpha_i) \prod_{j=1}^{JJ} (s_1 + \beta_j) \prod_{k=1}^{KK} (s_2 + \gamma_k)} \Bigg|_{\substack{s_1 = s-u \\ s_2 = u}} \right\} du$$

$$= \frac{1}{\prod_{i=1}^{II} (s + \alpha_i)} \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \left\{ \frac{\mathrm{N_2}(s-u, u)}{\prod_{j=1}^{JJ} (s - u + \beta_j) \prod_{k=1}^{KK} (u + \gamma_k)} \right\} du$$

Since $\mathrm{N}_2(s_1, s_2)$ is composed of the same three types of factors as may be present in the denominator, therefore the substitution of $s-u$ for s_1 and u for s_2 will produce a separable integrand; i.e. the equation will be of the form

(9) $$\mathrm{F}_2(s) = \frac{\mathscr{N}(s)}{\mathscr{D}(s)} \sum_i \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\overline{\mathscr{N}}_i(s-u)}{\overline{\mathscr{D}}(s-u)} \frac{\tilde{\mathscr{N}}}{\tilde{\mathscr{D}}} du$$

and the integrand can be evaluated by summing the residues of the integrand at the zeroes of $\tilde{\mathscr{D}}(u)$. Equation (7) thus becomes

(10)
$$\mathrm{F}_2(s) = \frac{1}{\prod_{i=1}^{II}(s+\alpha_i)} \sum_{\substack{\text{Residues}\\ \text{at}\\ u=-\gamma_k}} \left\{ \frac{\mathrm{N}_2(s-u,u)}{\prod_{j=1}^{JJ}(s-u+\beta_j)\prod_{k=1}^{KK}(u+\gamma_k)} \right\}$$
$$= \sum_{K=1}^{KK} \left\{ \frac{(u+\gamma_K)\mathrm{N}_2(s-u,u)}{\prod_{i=1}^{II}(s+\alpha_i)\prod_{j=1}^{JJ}(s-u+\beta_j)\prod_{k=1}^{KK}(u+\gamma_k)} \Big|_{u=-\gamma_K} \right\}$$

This expression is a function of only one Laplace variable and therefore can be expanded using Heaviside's theorems [4, pp. 317-323], yielding

(11)
$$\mathrm{F}_2(s) = \sum_{J=1}^{JJ} \sum_{K=1}^{KK} \left\{ \frac{ \frac{(s-u+\beta_J)(u+\gamma_K)\mathrm{N}_2(s-u,u)}{\prod_{i=1}^{II}(s+\alpha_i)\prod_{j=1}^{JJ}(s-u+\beta_j)\prod_{k=1}^{KK}(u+\gamma_k)} \Big|_{\substack{u=-\gamma_K\\ s=-\gamma_K-\beta_J}} }{(s+\gamma_k+\beta_J)} \right\}$$
$$+ \sum_{I=1}^{II} \left\{ \frac{ \frac{(s+\alpha_I)(u+\gamma_K)\mathrm{N}_2(s-u,u)}{\prod_{i=1}^{II}(s+\alpha_i)\prod_{j=1}^{JJ}(s-u+\beta_j)\prod_{k=1}^{KK}(u+\gamma_k)} \Big|_{\substack{u=-\gamma_K\\ s=-\alpha_I}} }{(s+\alpha_I)} \right\}$$

The variables s and u in the bracketed expressions are defined by the substitutions in (8), of $s-u$ for s_1 and u for s_2. The expressions can now be written in terms of the original variables by replacing u with s_2 and s with s_1+s_2, yielding

$$\mathrm{F}_2(s)=\sum_{J=1}^{JJ}\sum_{K=1}^{KK}\left\{\frac{\frac{(s_1+\beta_J)(s_2+\gamma_K)\mathrm{N}_2(s_1,s_2)}{\prod_{i=1}^{II}(s_1+s_2+\alpha_i)\prod_{j=1}^{JJ}(s_1+\beta_j)\prod_{k=1}^{KK}(s_2+\gamma_k)}\Big|_{\substack{s_1=-\beta_J\\ s_2=-\gamma_K}}}{(s+\gamma_k+\beta_J)}\right\}$$

(12)

$$+\sum_{I=1}^{II}\left\{\frac{\sum_{K=1}^{KK}\left[\frac{(s_1+s_2+\alpha_I)(s_2+\gamma_K)\mathrm{N}_2(s_1,s_2)}{\prod_{i=1}^{II}(s_1+s_2+\alpha_I)\prod_{j=1}^{JJ}(s_1+\beta_j)\prod_{k=1}^{KK}(s_2+\gamma_k)}\Big|_{\substack{s_1=-\gamma_K-\alpha_I\\ s_2=-\gamma_K}}\right]}{(s+\alpha_I)}\right\}$$

Comparing this with (7) reveals that each term of the summation contains $\mathrm{F}_2(s_1,s_2)$. Equation (12) can therefore be simplified to the desired result:

$$\mathrm{F}_2(s)=\sum_{J=1}^{JJ}\sum_{K=1}^{KK}\left\{\frac{(s_1+\beta_J)(s_2+\gamma_K)\mathrm{F}_2(s_1,s_2)\big|_{\substack{s_1=-\beta_J\\ s_2=-\gamma_K}}}{(s+\gamma_k+\beta_J)}\right\}$$

(13)

$$+\sum_{I=1}^{II}\left\{\frac{\sum_{K=1}^{KK}\left[(s_1+s_2+\alpha_I)(s_2+\gamma_K)\mathrm{F}_2(s_1,s_2)\big|_{\substack{s_1=-\gamma_K-\alpha_I\\ s_2=-\gamma_K}}\right]}{(s+\alpha_I)}\right\}$$

THEOREM 1. *The reduction and expansion of a realizable two-dimensional Laplace transform kernel of the form*

(14)
$$\mathrm{F}_2(s_1,s_2)=\frac{\mathrm{N}_2(s_1,s_2)}{\prod_{i=1}^{II}(s_1+s_2+\alpha_i)\prod_{j=1}^{JJ}(s_1+\beta_j)\prod_{k=1}^{KK}(s_2+\gamma_k)}$$

Where $\mathrm{N}_2(s_1,s_2)$ *is a polynomial in* s_1 *and* s_2 *, yields*

$$\mathrm{F}_2(s)=\sum_{J=1}^{JJ}\sum_{K=1}^{KK}\left\{\frac{(s_1+\beta_J)(s_2+\gamma_K)\mathrm{F}_2(s_1,s_2)\big|_{\substack{s_1=-\beta_J\\ s_2=-\gamma_K}}}{(s+\gamma_k+\beta_J)}\right\}$$

(15)

$$+\sum_{I=1}^{II}\left\{\frac{\sum_{K=1}^{KK}\left[(s_1+s_2+\alpha_I)(s_2+\gamma_K)\mathrm{F}_2(s_1,s_2)\big|_{\substack{s_1=-\gamma_K-\alpha_I\\ s_2=-\gamma_K}}\right]}{(s+\alpha_I)}\right\}$$

Note that the realizability criterion is imposed in order to assure that the association reduction would not yield a one-dimensional kernel having a numerator of higher degree than the denominator, which would mean that the resulting kernel is not Heaviside expandable.
Note that the order in which the association reduction integrals are evaluated is arbitrary, and therefore the roles of the variables s_1 and s_2 in Theorem 1 could be interchanged, if such is deemed beneficial. This will be, of course, equally valid for all n variables in the reduction of an n-dimensional kernel.

Example. Consider the reduction and expansion of a typical two-dimensional kernel

(16) $$\mathrm{F}_2(s_1,s_2)=\frac{K(s_1+\delta)}{(s_1+s_2+\beta)(s_1+\gamma)(s_1+\alpha)(s_2+\alpha)}$$

Using the above theorem,

(17) $$\begin{aligned}\mathrm{F}_2(s)=&\frac{\left[(s_1+\gamma)(s_2+\gamma)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=-\gamma\\ s_2=-\gamma}}\right]}{s+2\gamma}\\&+\frac{\left[(s_1+\alpha)(s_2+\alpha)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=-\alpha\\ s_2=-\alpha}}\right]}{s+2\alpha}\\&+\frac{\left[\begin{array}{l}(s_1+\alpha)(s_2+\gamma)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=-\alpha\\ s_2=-\gamma}}\\+(s_1+\gamma)(s_2+\alpha)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=-\gamma\\ s_2=-\alpha}}\end{array}\right]}{s+\alpha+\gamma}\\&+\frac{\left[\begin{array}{l}(s_1+s_2+\beta)(s_2+\gamma)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=\gamma-\beta\\ s_2=-\gamma}}\\+(s_1+s_2+\beta)(s_2+\alpha)\mathrm{F}_2(s_1,s_2)|_{\substack{s_1=\alpha-\beta\\ s_2=-\alpha}}\end{array}\right]}{s+\beta}\end{aligned}$$

Plugging in the indicated terms yields

(18) $$\begin{aligned}\mathrm{F}_2(s)=&\frac{\left[\dfrac{K(\delta-\gamma)}{(\beta-2\gamma)(\alpha-\gamma)^2}\right]}{(s+2\gamma)}+\frac{\left[\dfrac{K(\delta-\alpha)}{(\beta-2\alpha)(\gamma-\alpha)^2}\right]}{(s+2\alpha)}\\&+\frac{\left[\dfrac{-K(2\delta-\alpha-\gamma)}{(\beta-\alpha-\gamma)(\gamma-\alpha)^2}\right]}{(s+\alpha+\gamma)}+\frac{\left[\dfrac{2K(2\delta-\beta)}{(\alpha+\gamma-\beta)(2\gamma-\beta)(2\alpha-\beta)}\right]}{(s+\beta)}\end{aligned}$$

This is much easier than direct association reduction followed by partial fraction expansion.

Part 2 – Reduction of Order Via Association of Variables [5]

In certain types of systems analysis, it becomes essential to invert the n-dimensional Laplace transform and specify the inverse image at a single variable, t. We denote this image function of one variable as

(1) $\mathrm{g}(\mathrm{t}) = \mathrm{f}(t_1, t_2, \cdots, t_n)|_{t_1 = t_2 = \cdots = t_n = t}$

One approach to obtain the time function, $\mathrm{g}(\mathrm{t})$, is to associate with $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n)$ a function $\mathrm{G}(\mathrm{s})$ from which an application of the one-dimensional inverse transform yields $\mathrm{g}(\mathrm{t})$. This particular approach is called "Association of Variables". The function $\mathrm{G}(\mathrm{s})$ is said to be the associated transform of $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n)$.

Suppose $\mathrm{G}(\mathrm{s})$ is the associated transform of $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n)$ and $\mathrm{G}_1(\mathrm{s})$ is that of $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_{m-1}, s_{m+1}, \cdots, s_n)$, $m \le n$. Let k be any constant, and we restrict the variables $s, s_1, s_2, \cdots, s_n$ to the right half of the complex plane.

<u>Theorem 1</u> If a given function $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n)$ can be written in the form

$$\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n) = \frac{k}{s_m (s_m + a)} \mathrm{F}_1(\mathrm{s}_1, s_2, \cdots, s_{m-1}, s_{m+1}, \cdots, s_n)$$

and if $\mathrm{F}_1(\mathrm{s}_1, s_2, \cdots, s_{m-1}, s_{m+1}, \cdots, s_n) \xrightarrow{\mathrm{A}_{n-1}} \mathrm{G}_1(s)$. Then the associated transform

$$\mathrm{F}_1(\mathrm{s}_1, s_2, \cdots, s_n) \xrightarrow{\mathrm{A}_n} \mathrm{G}(\mathrm{s}) = \frac{k}{a}[\mathrm{G}_1(s) - \mathrm{G}_1(s+a)],$$

Where A_n means the association process for finding $\mathrm{G}(\mathrm{s})$ from $\mathrm{F}(\mathrm{s}_1, s_2, \cdots, s_n)$ and $\mathrm{A}_{\mathrm{n}-1}$ has a similar meaning. (Proof: see [5]).

Example Consider $\mathrm{F}(\mathrm{s}_1, s_2, s_3) = \dfrac{k}{s_3(s_1 + a)(s_2 + b)(s_3 + c)}$

And let $\mathrm{F}_1(s_1, s_2) = \dfrac{1}{(s_1 + a)(s_2 + b)}$

Using the table given in [6] $\mathrm{F}_1(s_1, s_2) \xrightarrow{\mathrm{A}_2} \mathrm{G}_1(s) = \dfrac{1}{s + a + b}$.

Therefore, according to Theorem 1,

$$\mathrm{F}(\mathrm{s}_1, s_2, s_3) \xrightarrow{\mathrm{A}_3} \mathrm{G}(\mathrm{s}) = \frac{k}{c}\left[\frac{1}{(s + a + b)} + \frac{1}{(s + a + b + c)}\right]$$

$$= \frac{k}{(s + a + b)(s + a + b + c)}$$

Conclusions

Although the field of multidimensional Laplace transforms does not appear to be a heavily-worked one, still, the above is only a tiny sampling of the theorems to be found in the literature. The understanding of this subject by humanity is still far from good.

References

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