

# Square Wave Signals

## Chapter 7 - Mixed-Frequency AC Signals

---

It has been found that *any* repeating, non-sinusoidal waveform can be equated to a combination of DC voltage, sine waves, and/or cosine waves (sine waves with a 90 degree phase shift) at various amplitudes and frequencies. This is true no matter how strange or convoluted the waveform

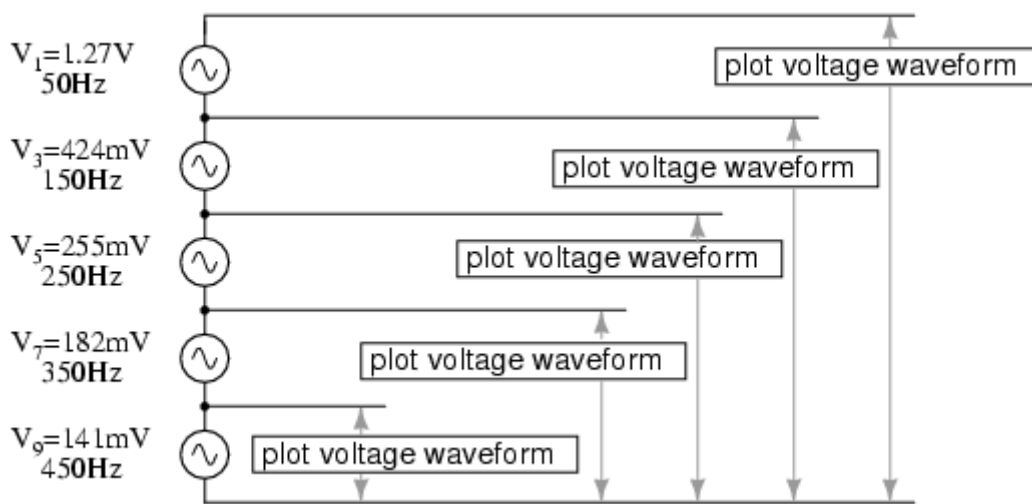
(<https://www.allaboutcircuits.com/textbook/alternating-current/chpt-1/ac-waveforms/>) in question may be. So long as it repeats itself regularly over time, it is reducible to this series of sinusoidal waves. In particular, it has been found that square waves are mathematically equivalent to the sum of a sine wave at that same frequency, plus an infinite series of odd-multiple frequency sine waves at diminishing amplitude:

1 V (peak) repeating square wave at 50 Hz is equivalent to:

$$\begin{aligned} & \left(\frac{4}{\pi}\right) (1 \text{ V peak sine wave at } 50 \text{ Hz}) \\ & + \left(\frac{4}{\pi}\right) (1/3 \text{ V peak sine wave at } 150 \text{ Hz}) \\ & + \left(\frac{4}{\pi}\right) (1/5 \text{ V peak sine wave at } 250 \text{ Hz}) \\ & + \left(\frac{4}{\pi}\right) (1/7 \text{ V peak sine wave at } 350 \text{ Hz}) \\ & + \left(\frac{4}{\pi}\right) (1/9 \text{ V peak sine wave at } 450 \text{ Hz}) \\ & + \dots \textit{ad infinitum} \dots \end{aligned}$$

This truth about waveforms at first may seem too strange to believe. However, if a square wave is actually an infinite series of sine wave harmonics added together, it stands to reason that we should be able to prove this by adding together several sine wave harmonics to produce a close approximation of a square wave. This reasoning is not only sound, but easily demonstrated with SPICE.

The circuit we'll be simulating is nothing more than several sine wave AC voltage sources of the proper amplitudes and frequencies connected together in series. We'll use SPICE to plot the voltage waveforms across successive additions of voltage sources, like this: (Figure below)

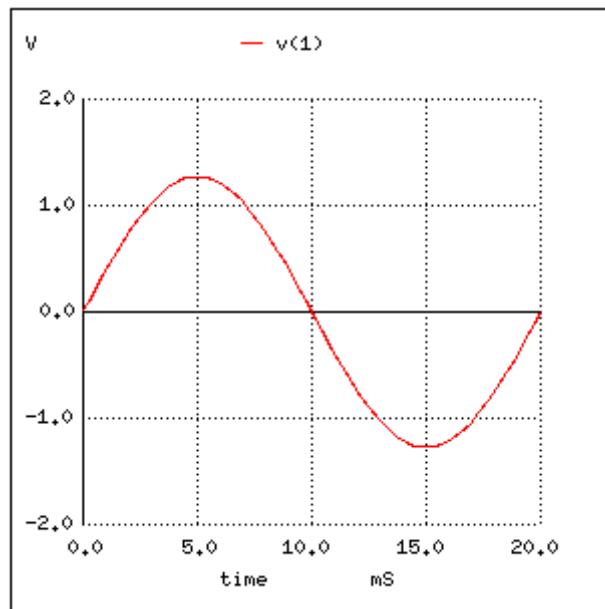


*A square wave is approximated by the sum of harmonics.*

In this particular SPICE simulation, I've summed the 1st, 3rd, 5th, 7th, and 9th harmonic voltage sources in series for a total of five AC voltage sources. The fundamental frequency is 50 Hz and each harmonic is, of course, an integer multiple of that frequency. The amplitude (voltage) figures are not random numbers; rather, they have been arrived at through the equations shown in the frequency series (the fraction  $4/\pi$  multiplied by 1, 1/3, 1/5, 1/7, etc. for each of the increasing odd harmonics).

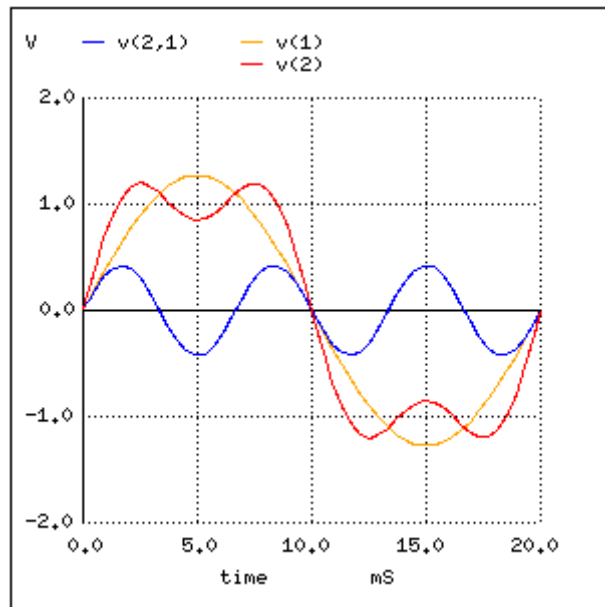
```
building a squarewave
v1 1 0 sin (0 1.27324 50 0 0)      1st harmonic (50 Hz)
v3 2 1 sin (0 424.413m 150 0 0)    3rd harmonic
v5 3 2 sin (0 254.648m 250 0 0)    5th harmonic
v7 4 3 sin (0 181.891m 350 0 0)    7th harmonic
v9 5 4 sin (0 141.471m 450 0 0)    9th harmonic
r1 5 0 10k
.tran 1m 20m
.plot tran v(1,0)      Plot 1st harmonic
.plot tran v(2,0)      Plot 1st + 3rd harmonics
.plot tran v(3,0)      Plot 1st + 3rd + 5th harmonics
.plot tran v(4,0)      Plot 1st + 3rd + 5th + 7th harmonics
.plot tran v(5,0)      Plot 1st + . . . + 9th harmonics
.end
```

I'll narrate the analysis step by step from here, explaining what it is we're looking at. In this first plot, we see the fundamental-frequency sine-wave of 50 Hz by itself. It is nothing but a pure sine shape, with no additional harmonic content. This is the kind of waveform produced by an ideal AC power source: (Figure below)



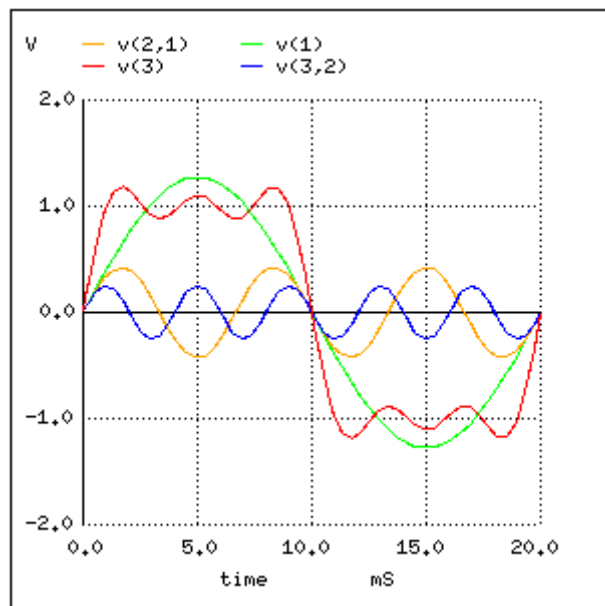
*Pure 50 Hz sinewave.*

Next, we see what happens when this clean and simple waveform is combined with the third harmonic (three times 50 Hz, or 150 Hz). Suddenly, it doesn't look like a clean sine wave any more: (Figure below)



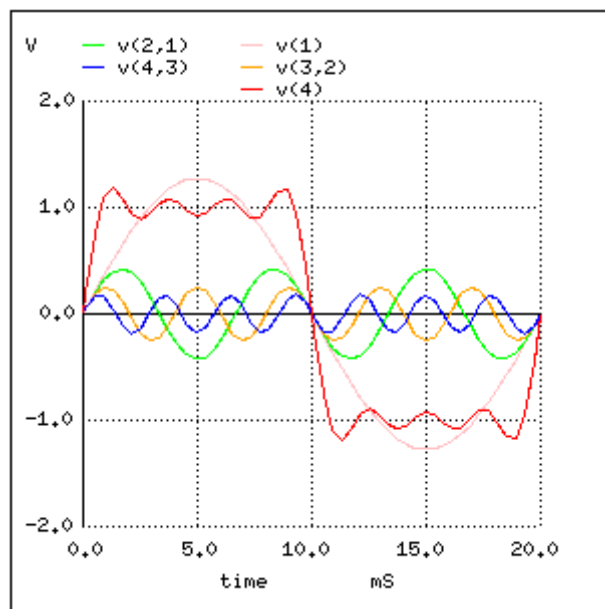
*Sum of 1st (50 Hz) and 3rd (150 Hz) harmonics approximates a 50 Hz square wave.*

The rise and fall times between positive and negative cycles are much steeper now, and the crests of the wave are closer to becoming flat like a squarewave. Watch what happens as we add the next odd harmonic frequency: (Figure below)



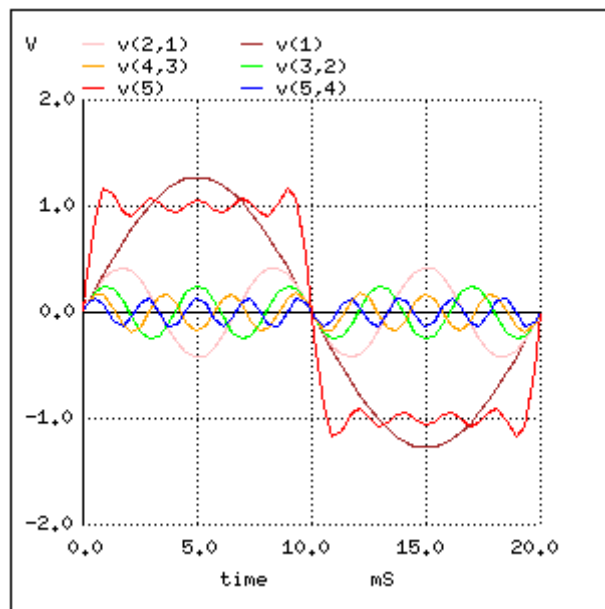
*Sum of 1st, 3rd and 5th harmonics approximates square wave.*

The most noticeable change here is how the crests of the wave have flattened even more. There are more several dips and crests at each end of the wave, but those dips and crests are smaller in amplitude than they were before. Watch again as we add the next odd harmonic waveform to the mix: (Figure below)



*Sum of 1st, 3rd, 5th, and 7th harmonics approximates square wave.*

Here we can see the wave becoming flatter at each peak. Finally, adding the 9th harmonic, the fifth sine wave voltage source in our circuit, we obtain this result: (Figure below)



*Sum of 1st, 3rd, 5th, 7th and 9th harmonics approximates square wave.*

The end result of adding the first five odd harmonic waveforms together (all at the proper amplitudes, of course) is a close approximation of a square wave. The point in doing this is to illustrate how we can build a square wave up from multiple sine waves at different frequencies, to prove that a pure square wave is actually equivalent to a *series* of sine waves. When a square wave AC voltage is applied to a circuit with reactive components (capacitors and inductors), those components react as if they were being exposed to several sine wave voltages of different frequencies, which in fact they are.

The fact that repeating, non-sinusoidal waves are equivalent to a definite series of additive DC voltage, sine waves, and/or cosine waves is a consequence of how waves work: a fundamental property of all wave-related phenomena, electrical or otherwise. The mathematical process of reducing a non-sinusoidal wave into these constituent frequencies is called *Fourier analysis*, the details

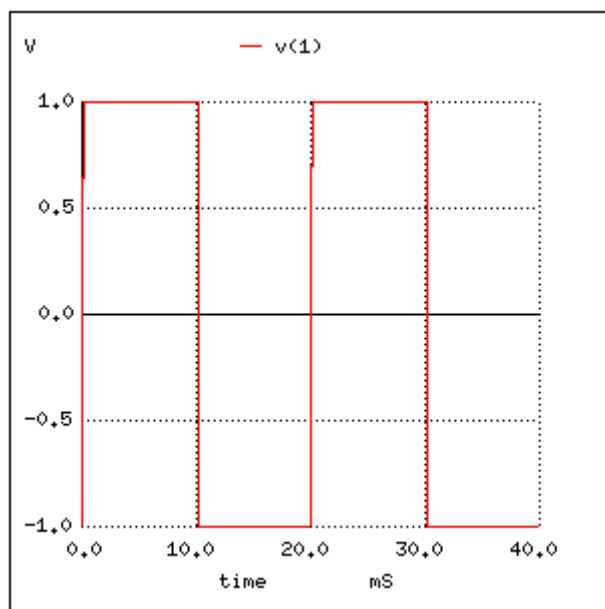
of which are well beyond the scope of this text. However, computer algorithms have been created to perform this analysis at high speeds on real waveforms, and its application in AC power quality and signal analysis is widespread.

SPICE has the ability to sample a waveform and reduce it into its constituent sine wave harmonics by way of a *Fourier Transform* algorithm, outputting the frequency analysis as a table of numbers. Let's try this on a square wave, which we already know is composed of odd-harmonic sine waves:

```
squarewave analysis netlist
v1 1 0 pulse (-1 1 0 .1m .1m 10m 20m)
r1 1 0 10k
.tran 1m 40m
.plot tran v(1,0)
.four 50 v(1,0)
.end
```

The *pulse* option in the netlist line describing voltage source v1 instructs SPICE to simulate a square-shaped “pulse” waveform, in this case one that is symmetrical (equal time for each half-cycle) and has a peak amplitude of 1 volt. First we'll plot the square wave to be analyzed: (Figure below)

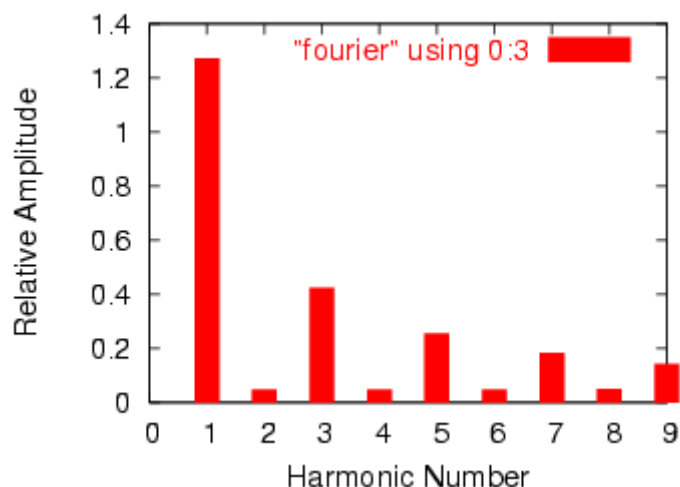




### *Squarewave for SPICE Fourier analysis*

Next, we'll print the Fourier analysis generated by SPICE for this square wave:

```
fourier components of transient response v(1)
dc component = -2.439E-02
harmonic  frequency  fourier    normalized  phase    normalized
no         (hz)      component  component  (deg)    phase (deg)
1          5.000E+01  1.274E+00  1.000000   -2.195   0.000
2          1.000E+02  4.892E-02  0.038415  -94.390  -92.195
3          1.500E+02  4.253E-01  0.333987   -6.585   -4.390
4          2.000E+02  4.936E-02  0.038757  -98.780  -96.585
5          2.500E+02  2.562E-01  0.201179  -10.976   -8.780
6          3.000E+02  5.010E-02  0.039337  -103.171 -100.976
7          3.500E+02  1.841E-01  0.144549  -15.366  -13.171
8          4.000E+02  5.116E-02  0.040175  -107.561 -105.366
9          4.500E+02  1.443E-01  0.113316  -19.756  -17.561
total harmonic distortion = 43.805747 percent
```



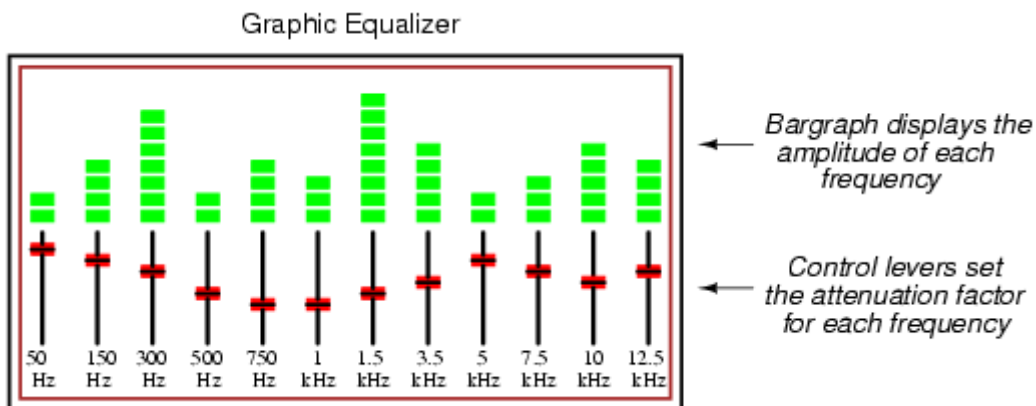
*Plot of Fourier analysis results.*

Here, (Figure above) SPICE has broken the waveform down into a spectrum of sinusoidal frequencies up to the ninth harmonic, plus a small DC voltage labelled DC component. I had to inform SPICE of the fundamental frequency (for a square wave with a 20 millisecond period, this frequency is 50 Hz), so it knew how to classify the harmonics. Note how small the figures are for all the even harmonics (2nd, 4th, 6th, 8th), and how the amplitudes of the odd harmonics diminish (1st is largest, 9th is smallest).

This same technique of “Fourier Transformation” is often used in computerized power instrumentation, sampling the AC waveform(s) and determining the harmonic content thereof. A common computer algorithm (sequence of program steps to perform a task) for this is the *Fast Fourier Transform* or *FFT* function. You need not be concerned with exactly how these computer routines work, but be aware of their existence and application.

This same mathematical technique used in SPICE to analyze the harmonic content of waves can be applied to the technical analysis of music: breaking up any particular sound into its constituent sine-

wave frequencies. In fact, you may have already seen a device designed to do just that without realizing what it was! A *graphic equalizer* is a piece of high-fidelity stereo equipment that controls (and sometimes displays) the nature of music's harmonic content. Equipped with several knobs or slide levers, the equalizer is able to selectively attenuate (reduce) the amplitude of certain frequencies present in music, to “customize” the sound for the listener's benefit. Typically, there will be a “bar graph” display next to each control lever, displaying the amplitude of each particular frequency. (Figure below)



*Hi-Fi audio graphic equalizer.*

A device built strictly to display—not control—the amplitudes of each frequency range for a mixed-frequency signal is typically called a *spectrum analyzer*. The design of spectrum analyzers may be as simple as a set of “filter” circuits (see the next chapter for details) designed to separate the different frequencies from each other, or as complex as a special-purpose digital computer running an FFT algorithm to mathematically split the signal into its harmonic components. Spectrum analyzers are often designed to analyze extremely high-frequency signals, such as those produced